

PROFILE OF SOLUTIONS FOR NONLOCAL EQUATIONS WITH CRITICAL AND SUPERCRITICAL NONLINEARITIES

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ABSTRACT. We study the restricted fractional laplacian problem

$$(I_\varepsilon) \left\{ \begin{array}{l} (-\Delta)^s u = u^p - \varepsilon u^q \quad \text{in } \Omega, \\ u \in H^s(\Omega) \cap L^{q+1}(\Omega), \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{array} \right.$$

where $s \in (0, 1)$, $q > p \geq \frac{N+2s}{N-2s}$ and $\varepsilon > 0$ is a parameter. Here $\Omega \subseteq \mathbb{R}^N$ is a bounded star-shaped domain with smooth boundary and $N > 2s$. We establish existence of a variational positive solution u_ε and characterise the asymptotic behaviour of u_ε as $\varepsilon \rightarrow 0$. When $p = \frac{N+2s}{N-2s}$, we describe how the solution u_ε concentrates and blows up at a interior point of the domain. Furthermore, we prove the *local uniqueness* of solution of the above problem when Ω is a convex symmetric domain of \mathbb{R}^N with $N > 4s$ and $p = \frac{N+2s}{N-2s}$.

1. Introduction

There has been considerable interest in understanding the asymptotic behavior of positive solutions of the elliptic problem

$$(1.1) \quad \left\{ \begin{array}{l} \varepsilon^{2s} (-\Delta)^s u = f(u) \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \mathbb{R}^N \setminus \Omega, \end{array} \right.$$

where $\varepsilon > 0$ is a parameter, $s \in (0, 1)$ and f is having superlinear nonlinearity with $f(0) = 0$. Ω is a smooth bounded domain in \mathbb{R}^N . The existence and asymptotic behavior of solutions to (1.1) depend crucially on the behavior of f near 0. It is easy to check that problem (1.1) admits solutions on Ω if $f'(0) < 0$, while there may not be any nontrivial solutions for small $\varepsilon > 0$ if $f'(0) > 0$. The case of $f'(0) < 0$ has been studied by many authors. To mention a few of them in the local case, we refer the papers [17], [24] and the references therein. In the nonlocal case, not much is known. Multi-peak solutions of a fractional Schrödinger equation in the whole of \mathbb{R}^N was considered in [14]. In [15], Dávila, et al constructed a family of solutions which have the properties that, when $\varepsilon \rightarrow 0$, those solutions concentrate at an interior point of the domain in the form of a scaling ground state in entire space. Bubble solutions for the fractional problems involving the almost subcritical or almost supercritical powers were considered in Dávila et al et al [13].

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In this paper, we consider the problem in the *zero mass case* i.e., when $f(0) = 0$ and $f'(0) = 0$. The problem (1.1) can be viewed as *borderline* problems. When $s = 1$, Berestycki and Lions in [3] proved the existence of ground state solutions if $f(u)$ behaves like $|u|^p$ for large u and $|u|^q$ for small u where p and q are respectively supercritical and subcritical.

In this paper, we consider the following family of problems:

$$(1.2) \quad \begin{cases} (-\Delta)^s u = u^p - \varepsilon u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u \in H^s(\Omega) \cap L^{q+1}(\Omega), \end{cases}$$

where $s \in (0, 1)$ is fixed, $(-\Delta)^s$ denotes the fractional Laplace operator defined, up to a normalisation factor, as

$$(1.3) \quad -(-\Delta)^s u(x) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy, \quad x \in \mathbb{R}^N.$$

In (1.2), $q > p \geq 2^* - 1 = \frac{N+2s}{N-2s}$, $\varepsilon > 0$ is a parameter, $\Omega \subseteq \mathbb{R}^N$ is a bounded star-shaped domain with smooth boundary and $N > 2s$. Note under a suitable change of variable (1.2) can be transformed in the form of (1.1).

We denote by $H^s(\Omega)$, the usual fractional Sobolev space endowed with the so-called Gagliardo norm

$$(1.4) \quad \|g\|_{H^s(\Omega)} = \|g\|_{L^2(\Omega)} + \left(\int_{\Omega \times \Omega} \frac{|g(x) - g(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2}.$$

For further details on the fractional Sobolev spaces we refer to [25] and to the references therein. Note that, in problem (1.2) the Dirichlet datum is given in $\mathbb{R}^N \setminus \Omega$ and not simply on $\partial\Omega$ and therefore we need to introduce a new functional space X_0 , which, in our opinion, is the suitable space to work with.

$$(1.5) \quad X_0(\Omega) := \{v \in H^s(\mathbb{R}^N) : v = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}.$$

By [30, Lemma 6 and 7], it follows that

$$(1.6) \quad \|v\|_{X_0} = \left(\int_Q \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2},$$

where $Q = \mathbb{R}^N \setminus (\Omega^c \times \Omega^c)$, is a norm on X_0 and $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space, with the inner product

$$\langle u, v \rangle_{X_0} = \int_Q \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy.$$

We observe that, norms in (1.4) and (1.6) are not same, since $\Omega \times \Omega$ is strictly contained in Q . Clearly, the integral in (1.6) can be extended to whole of \mathbb{R}^{2N} as $v = 0$ in $\mathbb{R}^N \setminus \Omega$. It follows from [30, Lemma 8] that the embedding $X_0 \hookrightarrow L^r(\mathbb{R}^N)$ is compact, for any $r \in [1, 2^*)$ and from [29, Lemma 9] that $X_0 \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is continuous.

Definition 1.1. We say that $u \in X_0 \cap L^{q+1}(\Omega)$ is a weak solution of Eq. (1.2), if $u > 0$ in Ω and for every $\varphi \in X_0$,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy = \int_{\Omega} u^p \varphi \, dx - \varepsilon \int_{\Omega} u^q \varphi \, dx.$$

In recent years, a great deal of attention has been devoted to equations of elliptic/parabolic type with fractional and non-local operators because these kind of equations play important role in the real world and many perfect techniques which have been developed by well-known mathematicians during the past decades can not be directly applied to the fractional case. These equations arise from models in physics, engineering (see [23]), optimisation and finance (see [12]), obstacle problem (see [31]), conformal geometry and minimal surface (see [7]) and many more, see for instance, [2, 34, 35] and the references therein.

Nonlinear nonlocal problems of the form $(-\Delta)^s u = f(u)$ were studied by many authors where $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is a certain function. Since it is almost impossible to describe all the works involving them, we explain only few of them, which are related to our problem. When $s = \frac{1}{2}$, Cabré and Tan [6] proved the existence of positive solutions for equations having subcritical nonlinearities, they also established regularity result and the symmetric property. In [29], Servadei and Valdinoci studied the Brezis-Nirenberg problem in the nonlocal case. More precisely, they considered the nonlinearity of the form $\lambda u + u^{2^*-1}$, with $\lambda > 0$. On the other hand, in [30] the same authors studied mountain-pass solutions for the equation with general integro-differential operator and with the nonlinearities of subcritical growth. In [4], first and second authors of this paper studied the equation in whole of \mathbb{R}^N with nonlinearities involving critical and supercritical growth. They established decay estimate of solution and the gradient of the solution at infinity and using that they prove nonexistence result via Pohozaev identity. In [10], Choi studied the nonlocal system of equations and [11] dealt with asymptotic behavior of solutions for the spectral fractional laplacian, when $f(u) = u^{2^*-1} + \varepsilon u$ with $\varepsilon > 0$.

In the local case, $s = 1$, Merle and Peletier [22] considered the equation (1.2). They proved that for $N \geq 3$, problem (1.2) possesses a family of solutions concentrating at a point ξ_0 , which is a critical point of the Robin function R . In this paper we extend the result to the restricted fractional case.

For the supercritical case ($p > 2^* - 1$), define,

$$(1.7) \quad F(u, \Omega) = \frac{1}{2} \frac{\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy}{\int_{\Omega} |u|^{p+1} dx} + \frac{1}{q+1} \frac{\int_{\Omega} |u|^{q+1} dx}{\left(\int_{\Omega} |u|^{p+1} dx \right)^l},$$

where $l = \frac{2s(q+1) - N(p-1)}{2s(p+1) - N(p-1)}$, $u \in X_0(\Omega) \cap L^{q+1}(\Omega)$ and

$$(1.8) \quad \mathcal{K} := \inf \left\{ F(u, \mathbb{R}^N) : u \in H^s(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^{p+1} = 1 \right\}.$$

For the critical case ($p = 2^* - 1$), we consider the usual functional

$$(1.9) \quad S(u) = \frac{\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy}{\left(\int_{\Omega} |u|^{p+1} dx \right)^{\frac{2}{p+1}}},$$

where $u \in X_0(\Omega)$.

Define, the Sobolev constant

$$(1.10) \quad \mathcal{S} : = \inf_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx \right)^{\frac{2}{2^*}}}$$

or, equivalently,

$$\mathcal{S} = \inf \left\{ S(v) : v \in H^s(\mathbb{R}^N), \int_{\mathbb{R}^N} |v|^{2^*} dx = 1 \right\}.$$

It is well known that (see [11]), \mathcal{S} is achieved by

$$(1.11) \quad U(x) = c_{N,s} (1 + |x|^2)^{-\left(\frac{N-2s}{2}\right)},$$

where

$$(1.12) \quad c_{N,s} = 2^{\frac{N-2s}{2}} \left(\frac{\Gamma(\frac{N+2s}{2})}{\Gamma(\frac{N-2s}{2})} \right)^{\frac{N-2s}{4s}}.$$

A direct computation implies that for all $\varepsilon > 0$ and for any $a \in \mathbb{R}^N$, U is the unique solution satisfying

$$U_{\varepsilon,a}(x) = \varepsilon^{-\frac{N-2s}{2}} U\left(\frac{x-a}{\varepsilon}\right)$$

and verifies the following equation

$$(1.13) \quad \begin{cases} (-\Delta)^s U = U^{2^*-1} & \text{in } \mathbb{R}^N, \\ U > 0 & \text{in } \mathbb{R}^N, \\ U \in H^s(\mathbb{R}^N). \end{cases}$$

See [8] and [21].

Define the Green's function $G = G(x, y)$ of the operator $(-\Delta)^s$ in Ω for $x, y \in \Omega$ as

$$(1.14) \quad \begin{cases} (-\Delta_x)^s G(x, y) = \delta_y & \text{in } \Omega, \\ G(x, y) = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

It is convenient to introduce the regular part of G , which is often denoted by H , defined by

$$(1.15) \quad G(x, y) := F(x, y) - H(x, y),$$

where the function H satisfies

$$(1.16) \quad \begin{cases} (-\Delta_x)^s H(x, y) = 0 & \text{in } \Omega, \\ H(x, y) = F(x, y) & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

for any fixed $y \in \Omega$ and

$$(1.17) \quad F(x, y) = \frac{a_{N,s}}{|x - y|^{N-2s}},$$

is the fundamental solution of the elliptic operator $(-\Delta)^s$. In (1.17), $a_{N,s}$ is

$$a_{N,s} := \frac{\Gamma(\frac{N}{2} - s)}{2^{2s} \pi^{\frac{N}{2}} \Gamma(s)},$$

(see [5]). Define the Robin function as

$$(1.18) \quad R(x) = H(x, x).$$

For the continuity of R , see Abatangelo [1].

Definition 1.2. We say Ω is strictly star-shaped with respect to the point y , if

$$\langle x - y, n(x) \rangle > 0 \quad \forall \quad x \in \partial\Omega,$$

where $n(x)$ is the unit outward normal to $\partial\Omega$ at x .

We recall here the general Pohozaev identity in the nonlocal case due to Ros-Oton and Serra [26]: Let u be a bounded solution of

$$(1.19) \quad \begin{cases} (-\Delta)^s u = f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded $C^{1,1}$ domain, f is locally Lipschitz and $d(x) = \text{dist}(x, \partial\Omega)$. Then u satisfies the following identity:

$$(1.20) \quad (2s - N) \int_{\Omega} u f(u) \, dx + 2N \int_{\Omega} F(u) \, dx = \Gamma(1 + s)^2 \int_{\partial\Omega} \left(\frac{u(x)}{d^s(x)} \right)^2 \langle x, \nu(x) \rangle dS(x),$$

where $F(t) = \int_0^t f(s) ds$, $\nu(x)$ is the unit outward normal to $\partial\Omega$ at x and Γ is the Gamma function.

Translating the function u , it is easy to see that, when Ω is a $C^{1,1}$ bounded domain, the following general identity holds:

$$(1.21) \quad (2s - N) \int_{\Omega} u f(u) \, dx + 2N \int_{\Omega} F(u) \, dx = \Gamma(1 + s)^2 \int_{\partial\Omega} \left(\frac{u(x)}{d^s(x)} \right)^2 \langle x - y, \nu(x) \rangle dS(x),$$

for every $y \in \mathbb{R}^N$.

Note that, by the above Pohozaev identity (1.2) does not have any solution in a star-shaped domain when $\varepsilon = 0$.

We turn now to a brief description of the results presented below.

Theorem 1.1. *There exists $\varepsilon_n > 0$ and $\lambda_n > 0$ with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and λ_n uniformly bounded above and away from zero, such that*

- (i) *there exists a solution u_n to Eq. (1.2) corresponding to $\varepsilon = \varepsilon_n$;*
- (ii) *if $p > 2^* - 1$, then $F(\lambda_n u_n) \rightarrow \mathcal{K}$ and $\int_{\Omega} u_n^{p+1} dx \rightarrow 0$ as $n \rightarrow \infty$;*
- (iii) *if $p = 2^* - 1$, then $S(u_n) \rightarrow \mathcal{S}$ as $n \rightarrow \infty$ and there exist constants $A, B > 0$ such that for all $n \geq 1$, it holds $A < \int_{\Omega} u_n^{p+1} dx < B$,*

where $F(\cdot)$, $S(\cdot)$, \mathcal{K} and \mathcal{S} are defined as in (1.7), (1.9), (1.8) and (1.10) respectively.

Theorem 1.2. *Let Ω be a smooth bounded star-shaped domain with respect to 0, $2^* - 1 = p < q$. Suppose $u_\varepsilon \in X_0(\Omega)$ is a solution of Eq. (1.2) such that*

$$(1.22) \quad S(u_\varepsilon) \rightarrow \mathcal{S} \quad \text{and} \quad A < \int_{\Omega} u_\varepsilon^{p+1} dx < B,$$

where $S(\cdot)$, \mathcal{S} are as in (1.9) and (1.10) respectively. Let x_ε be a point such that $\|u_\varepsilon\|_{L^\infty} = u_\varepsilon(x_\varepsilon)$. Assume that, up to a subsequence $x_\varepsilon \rightarrow x_0$ as $\varepsilon \rightarrow 0$. Then x_0 is an interior point of Ω and along a subsequence

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \|u_\varepsilon\|_{L^\infty}^{q-p+2} = \frac{\omega_N c_{N,s}^{2^*}}{2} \frac{(q+1)R_{N,s,x_0}}{q(N-2s) - (N+2s)} s^2 \Gamma(s)^2 B\left(\frac{N}{2}, s\right)^2 \times \\ B\left(\frac{N}{2}, \left(\frac{N-2s}{2}\right)q - s\right)^{-1},$$

where $c_{N,s}$ is defined in (1.12) and $B(a,b)$ is the Beta function defined by

$$(1.23) \quad B(a,b) = \int_0^\infty t^{a-1} (1+t)^{-a-b} dt.$$

Here

$$R_{N,s,x_0} = \int_{\partial\Omega} \left(\frac{G(x, x_0)}{d^s(x)} \right)^2 \langle x - x_0, \nu \rangle dS.$$

Furthermore,

$$(1.24) \quad \lim_{\varepsilon \rightarrow 0} \frac{u_\varepsilon(x) \|u_\varepsilon\|_{L^\infty}}{d^s(x)} = \frac{\omega_N c_{N,s}^{2^*}}{2} \frac{\Gamma(\frac{N}{2})\Gamma(s)}{\Gamma(\frac{N+2s}{2})} \frac{G(x, x_0)}{d^s(x)} \quad \text{in } C(\bar{\Omega} \setminus \{x_0\}),$$

where $G(x, x_0)$ is the Green function as defined in (1.14) and $d(x) = \text{dist}(x, \partial\Omega)$.

Remark 1.1. *Under a suitable modification to the Theorem 1.2, a similar blow-up type result for the equation with $(-\Delta)^s$ operator in a smooth bounded domain Ω with outside zero Dirichlet boundary condition can be obtained for the nonlinearity $f_1(u) = u^{2^*-1-\varepsilon}$ under the assumption*

$$\tilde{F}(u_\varepsilon) := \frac{\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x - y|^{N+2s}} dx dy}{\left(\int_{\Omega} |u|^{2^*-\varepsilon} dx \right)^{\frac{2}{2^*-\varepsilon}}} \rightarrow \mathcal{S} \quad \text{whenever } N > 2s$$

and for the nonlinearity $f_2(u) = u^{2^*-1} + \varepsilon u$ under the assumption

$$S(u_\varepsilon) \rightarrow \mathcal{S} \quad \text{whenever } N > 4s.$$

Concerning the *uniqueness* problem, the shape of the domain plays an important role and hence some assumptions on Ω is needed, see [19]. To prove uniqueness theorem, our assumption on the domain are the following:

- (A1) Ω is symmetric with respect to the hyperplanes $\{x_i = 0\}, i = 1, 2, \dots, N$.
- (A2) Ω is convex in the x_i directions, $i = 1, 2, \dots, N$.

Remark 1.2. By (A1), (A2) and in virtue of Felmer and Wang's result [18, Theorem 3.1], every solution u_ε of (1.2) is symmetric with respect to the hyperplanes $\{x_i = 0\}, i = 1, \dots, N$ and strictly decreasing in the x_i direction, $i = 1, \dots, N$. Therefore

$$\max_{x \in \Omega} u_\varepsilon(x) = u_\varepsilon(0).$$

Theorem 1.3. Let $2^* - 1 = p < q$ and Ω be smooth bounded star-shaped domain in \mathbb{R}^N with respect to 0, $N > 4s$, satisfying (A1) and (A2). Suppose u_ε and v_ε are two solutions of (1.2) with $\max_{x \in \Omega} u_\varepsilon = \max_{x \in \Omega} v_\varepsilon$ and satisfy (1.22). Then, there exists $\varepsilon_0 > 0$ such that $\forall \varepsilon \in (0, \varepsilon_0)$,

$$u_\varepsilon \equiv v_\varepsilon \quad \text{in } \Omega.$$

The rest of the paper is organised as follows. In Section 2, we prove Theorem 1.1. Section 3 deals with the proof of Theorem 1.2. Section 4 is devoted to the study of uniqueness result. The last section is the Appendix.

Remark 1.3. To our knowledge there exists only two papers dealing with the asymptotic behavior of solutions of the restricted fractional see [15] and [13].

Notations: Throughout this paper C denotes the generic constants which may vary from line to line. Below are few notations which we use throughout the paper:

- ω_N = surface measure of unit ball in \mathbb{R}^N ,
- $G(x, y)$ denotes the Green function of $(-\Delta)^s$ in Ω ,
- $B(., .)$ and $\Gamma(.)$ denote the Beta function and the Gamma function respectively.

2. Asymptotic behavior

Proposition 2.1. Let $2^* - 1 \leq p < q$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the problem

$$(2.1) \quad \begin{cases} (-\Delta)^s v = \lambda_\varepsilon v^p - \varepsilon v^q & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v(x) = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

admits a solution v_ε , with the property that

$$A < \lambda_\varepsilon < B,$$

for some constants $A, B > 0$, independent of n . In addition

(i) if $p > 2^* - 1$, then $F(v_\varepsilon) \rightarrow \mathcal{K}$ and $\int_\Omega v_\varepsilon^{p+1} dx \rightarrow 0$ as $\varepsilon \rightarrow 0$;

(ii) if $p = 2^* - 1$, then $S(v_\varepsilon) \rightarrow \mathcal{S}$ as $\varepsilon \rightarrow 0$ and $\int_\Omega v_\varepsilon^{p+1} dx = 1$,

where \mathcal{K} and \mathcal{S} are defined as in (1.8) and (1.10) respectively.

Proof. Let $\Omega_\varepsilon = \frac{1}{\varepsilon^{\frac{p-1}{2(q-p)}}} \Omega$ and $X_0(\Omega_\varepsilon) = \{w \in H^s(\mathbb{R}^N) : w = 0 \text{ in } \mathbb{R}^N \setminus \Omega_\varepsilon\}$.

Clearly $\Omega_\varepsilon \rightarrow \mathbb{R}^N$ as $\varepsilon \rightarrow 0$. Let us consider the manifold N_ε defined by:

$$N_\varepsilon = \left\{ w \in X_0(\Omega_\varepsilon) \cap L^{q+1}(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} w^{p+1} dx = 1 \right\}.$$

On N_ε , the functional F can be written as:

$$\begin{aligned} F(w) &= \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} dx dy + \frac{1}{q+1} \int_{\Omega_\varepsilon} w^{q+1} dx \\ (2.2) \quad &=: \hat{F}(w). \end{aligned}$$

For $p \geq 2^* - 1$, define

$$(2.3) \quad S_\varepsilon := \inf_{w \in N_\varepsilon} \hat{F}(w) = \inf_{w \in N_\varepsilon} F(w).$$

Let $\{w_{n,\varepsilon}\} \subset N_\varepsilon$ be a minimizing sequence for (2.3). Therefore, we have,

$$\hat{F}(w_{n,\varepsilon}) \rightarrow S_\varepsilon \text{ as } n \rightarrow \infty, \int_{\Omega_\varepsilon} w_{n,\varepsilon}^{p+1} dx = 1.$$

Proceeding as in [4, Theorem 1.5], we can show that there exists $w_\varepsilon \in X_0(\Omega_\varepsilon) \cap L^{q+1}(\Omega_\varepsilon)$ such that $w_{n,\varepsilon} \rightharpoonup w_\varepsilon$ in $X_0(\Omega_\varepsilon)$ and w_ε satisfies,

$$(-\Delta)^s w_\varepsilon = \lambda_\varepsilon w_\varepsilon^p - w_\varepsilon^q \text{ in } \Omega_\varepsilon \quad \text{and} \quad \hat{F}(w_\varepsilon) = S_\varepsilon.$$

This yields,

$$\lambda_\varepsilon = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_\varepsilon(x) - w_\varepsilon(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\Omega_\varepsilon} w_\varepsilon^{q+1} dx.$$

Since, $\hat{F}(w_\varepsilon) = S_\varepsilon$ we have, $2S_\varepsilon < \lambda_\varepsilon < (q+1)S_\varepsilon$. In Theorem A.1 (see Appendix), let $\rho = \varepsilon^{\frac{-(p-1)}{2(q-1)}}$, then N_ρ and S_ρ are exactly same as N_ε and S_ε defined here. Letting $\varepsilon \rightarrow 0$ we have,

$$(2.4) \quad S_\varepsilon \rightarrow \mathcal{K} \text{ if } p > 2^* - 1, \quad S_\varepsilon \rightarrow \frac{\mathcal{S}}{2} \text{ if } p = 2^* - 1.$$

Hence, there exists $\varepsilon_0 > 0$ and $A, B > 0$ such that $A < \lambda_\varepsilon < B$ for all $\varepsilon \in (0, \varepsilon_0)$. Using the transformation

$$v_\varepsilon(x) = \varepsilon^{-\frac{1}{(q-p)}} w_\varepsilon(\varepsilon^{-\frac{p-1}{2(q-p)}} x),$$

we observe that v_ε is a solution of (2.1). Moreover, $\int_{\Omega_\varepsilon} w_\varepsilon^{p+1} dx = 1$ implies

$$\int_{\Omega_\varepsilon} v_\varepsilon^{p+1} dx = \varepsilon^{\frac{p(N-2s)-(N+2s)}{2s(q-p)}}. \text{ Hence,}$$

$$\int_{\Omega} v_\varepsilon^{p+1} dx = 1 \text{ if } p = 2^* - 1$$

and

$$\int_{\Omega} v_\varepsilon^{p+1} dx \rightarrow 0 \text{ as } \varepsilon \rightarrow 0, \quad p > 2^* - 1.$$

A simple calculation yields

$$F(w_\varepsilon) = \hat{F}(w_\varepsilon) = F(v_\varepsilon) \quad \text{when } p > 2^* - 1,$$

where F and \hat{F} are defined as in (1.7) and (1.1). This along with (2.4) and the fact that $F(w_\varepsilon) = S_\varepsilon$ implies

$$F(v_\varepsilon) \rightarrow \mathcal{K} \text{ if } p > 2^* - 1.$$

Moreover when $p = 2^* - 1$,

$$\mathcal{S} \leq S(v_\varepsilon) \leq 2\hat{F}(v_\varepsilon, \Omega) = 2\hat{F}(w_\varepsilon, \Omega_\varepsilon) = 2S_\varepsilon \rightarrow \mathcal{S}.$$

Hence

$$S(v_\varepsilon) \rightarrow \mathcal{S} \quad \text{if } p = 2^* - 1.$$

This completes the proof. \square

Proof of Theorem 1.1: Let v_ε and λ_ε be as in Proposition 2.1. Define, $u_\varepsilon = \lambda_\varepsilon^{\frac{1}{p-1}} v_\varepsilon$. Then it is easy to see that u_ε satisfies

$$(-\Delta)^s u_\varepsilon = u_\varepsilon^p - \varepsilon \lambda_\varepsilon^{\frac{-(q-1)}{p-1}} u_\varepsilon \quad \text{in } \Omega.$$

Using the bounds on λ_ε from Proposition 2.1, we can conclude that there exist solutions u_n of problem (1.2) along a sequence $\{\varepsilon_n\}_{n \geq 1}$ of values ε which tends to 0 as $n \rightarrow \infty$. Set $\lambda_n := \lambda_{\varepsilon_n}^{\frac{-1}{p-1}}$. Thus, from Proposition 2.1 it follows

$$F(\lambda_n u_n) \rightarrow \mathcal{K} \quad \text{and} \quad \int_{\Omega} u_n^{p+1} \rightarrow 0 \quad \text{when } p > 2^* - 1$$

and

$$S(\lambda_n u_n) \rightarrow \mathcal{S} \quad \text{and} \quad A < \int_{\Omega} u_n^{p+1} < B \quad \text{when } p = 2^* - 1,$$

for some $A, B > 0$. Since $S(\lambda_n u_n) = S(u_n)$, theorem follows. \square

3. The case $p = 2^* - 1$ and the proof of Theorem 1.2

Lemma 3.1. *Let u_ε be as in Theorem 1.2. Then $\|u_\varepsilon\|_\infty \rightarrow \infty$ as $\varepsilon \rightarrow 0$.*

Proof. Since, u_ε is as in Theorem 1.2, we have

$$(3.1) \quad \int_{\Omega} u_\varepsilon^{2^*} dx = c,$$

where $c \in (A, B)$. Suppose, $\|u_\varepsilon\|_\infty$ is uniformly bounded. Therefore, by the Schauder estimate (see [28], [27]), $u_\varepsilon \rightarrow u$ in $C_{loc}^{2s-\delta}(\Omega) \cap C^{s-\delta}(\mathbb{R}^N)$, for any $\delta > 0$. By the definition of weak solution, we have

$$(3.2) \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_\varepsilon(x) - u_\varepsilon(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy = \int_{\Omega} (u_\varepsilon^{2^*-1} - \varepsilon u_\varepsilon^q) \varphi dx \quad \forall \varphi \in C_0^\infty(\Omega).$$

Moreover, as $\|u_\varepsilon\|_{C^s(\mathbb{R}^N)}$ is uniformly bounded (see [27, Proposition 1.1]), we get

$$\frac{(u_\varepsilon(x) - u_\varepsilon(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \leq C \frac{|x - y|^s |\nabla \varphi|_{L^\infty} |x - y|}{|x - y|^{N+2s}} \leq C \frac{1}{|x - y|^{N-1+s}}.$$

Therefore using the dominated convergence theorem, we can pass to the limit in (3.2) and get,

$$(3.3) \quad \begin{cases} (-\Delta)^s u = u^{2^*-1} & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where

$$(3.4) \quad A < \int_{\Omega} u^{2^*} dx < B.$$

As $A > 0$, the above expression implies u is a nontrivial solution in a bounded star-shaped domain. Since, $u \in C(\mathbb{R}^N)$ and $u = 0$ in $\mathbb{R}^N \setminus \Omega$, clearly u is a bounded solution. By the maximum principle ([31, Proposition 2.17]), we also have $u > 0$

in Ω . This gives a contradiction due to the Pohozaev identity [26, Corollary 1.3]. Hence the lemma follows. \square

Let x_ε be a local maximum point of u_ε and $\gamma_\varepsilon \in \mathbb{R}^+$ such that

$$(3.5) \quad u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|_\infty = \gamma_\varepsilon^{-\frac{N-2s}{2}}.$$

Then $\gamma_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Lemma 3.2. (*Blow-up at an interior point*) Let $x_0 := \lim_{\varepsilon \rightarrow 0} x_\varepsilon$, then x_0 is an interior point of Ω .

Proof. Let λ_1 be the first eigenvalue of $(-\Delta)^s$ in Ω and φ_1 be a corresponding eigenfunction (see [29]), that is, φ_1 satisfies

$$\begin{aligned} (-\Delta)^s \varphi_1 &= \lambda_1 \varphi_1 \quad \text{in } \Omega, \\ \varphi_1 &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{aligned}$$

Moreover, as u_ε is a classical solution of (1.2) (see [4, Proposition 3.1]), we have

$$\begin{aligned} \lambda_1 \int_{\Omega} \varphi_1 u_\varepsilon dx &= \int_{\Omega} (-\Delta)^s \varphi_1 u_\varepsilon dx = \int_{\Omega} \varphi_1 (-\Delta)^s u_\varepsilon dx \\ &= \int_{\Omega} \varphi_1 u_\varepsilon^{2^*-1} dx - \int_{\Omega} \varphi_1 u_\varepsilon^q dx \\ &\leq \int_{\Omega} \varphi_1 u_\varepsilon^{2^*-1} dx \\ &\leq \left(\int_{\Omega} u_\varepsilon^{2^*} dx \right)^{\frac{2^*-1}{2^*}} \left(\int_{\Omega} \varphi_1^{2^*} dx \right)^{\frac{1}{2^*}} \\ (3.6) \quad &\leq B^{\frac{2^*-1}{2^*}} \left(\int_{\Omega} \varphi_1^{2^*} dx \right)^{\frac{1}{2^*}} \leq C', \end{aligned}$$

for some constant C' . Hence $\int_{\Omega} \varphi_1 u_\varepsilon dx \leq \frac{C'}{\lambda_1}$. Since, $\varphi_1 \geq C$ on $\Omega' \subset\subset \Omega$, we obtain

$$(3.7) \quad \int_{\Omega'} u_\varepsilon \leq C(\Omega'),$$

for any $\Omega' \subset\subset \Omega$.

Define

$$O(\delta) := \{z \in \Omega : \text{dist}(z, \partial\Omega) < \delta\}$$

and

$$I(\delta) := \{z \in \Omega : \text{dist}(z, \partial\Omega) > \delta\}.$$

Claim: There exists $C > 0$ such that

$$\sup_{O(\delta)} u_\varepsilon(x) \leq C \quad \forall \varepsilon > 0.$$

We observe that $u_\varepsilon \in C^2(\bar{\Omega})$ (see [4, Theorem 1.2]). Therefore, if Ω is strictly convex, the moving plane argument, which is given in the proof of [6, Theorem 1.6] for $s = \frac{1}{2}$ can be extended to any $s \in (0, 1)$, yields the fact that each solution u_ε increases along an arbitrary straight line toward inside of Ω emanating from a point on $\partial\Omega$ (see for instance [11, Lemma 3.1]). Hence following an argument as in [20], we can find $\gamma, \delta > 0$ such that for any $x \in O(\delta)$, there exists a measurable set

Γ_x with (i) $\text{meas}(\Gamma_x) \geq \gamma$, (ii) $\Gamma_x \subset I(\frac{\delta}{2})$, and (iii) $u_\varepsilon(y) \geq u_\varepsilon(x)$ for any $y \in \Gamma_x$. In particular, Γ_x can be taken as a cone with vertex at x . Let $\Omega' = I(\frac{\delta}{2})$. Then for any $x \in O(\delta)$, we have

$$u_\varepsilon(x) \leq \frac{1}{\text{meas}(\Gamma_x)} \int_{\Gamma_x} u_\varepsilon(y) dy \leq \gamma^{-1} \int_{\Omega'} u_\varepsilon \leq C(\Omega').$$

This proves the claim when Ω is strictly convex. The general case can be proved using Kelvin transform in the extended domain (see, for instance, [20], [11], [10]).

From Lemma 3.1, we have $u_\varepsilon(x_\varepsilon) \rightarrow \infty$ as $\varepsilon \rightarrow 0$. On the other hand, the above claim implies u_ε is uniformly bounded near the boundary for all small $\varepsilon > 0$. Hence passing to a subsequence, the point x_ε converges to an interior point $x_0 \in \Omega$. \square

Define

$$(3.8) \quad z_\varepsilon(x) = \gamma_\varepsilon^{\frac{N-2s}{2}} u_\varepsilon(\gamma_\varepsilon x + x_\varepsilon).$$

Then $\|z_\varepsilon\|_\infty = 1$ and satisfies

$$(3.9) \quad \begin{cases} (-\Delta)^s z_\varepsilon = z_\varepsilon^{2^*-1} - \varepsilon \gamma_\varepsilon^{\frac{(N+2s)-q(N-2s)}{2}} z_\varepsilon^q & \text{in } \Omega_\varepsilon, \\ z_\varepsilon > 0 & \text{in } \Omega_\varepsilon, \\ z_\varepsilon = 0 & \text{in } \mathbb{R}^N \setminus \Omega_\varepsilon, \end{cases}$$

where $\Omega_\varepsilon = \frac{\Omega - x_\varepsilon}{\gamma_\varepsilon}$.

Lemma 3.3. *Suppose z_ε is as in (3.8). Then*

$$(i) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon \gamma_\varepsilon^{\frac{(N+2s)-q(N-2s)}{2}} = 0$$

(ii) *There exists $Z \in H^s(\mathbb{R}^N)$ such that $z_\varepsilon \rightarrow Z$ in $C_{loc}^{2s-\delta}(\mathbb{R}^N)$ as $\varepsilon \rightarrow 0$, for any $\delta > 0$.*

$$(iii) \quad Z \text{ satisfies Eq. (1.13) and } Z(x) = \left[1 + \frac{|x|^2}{\mu_{N,s}} \right]^{-\frac{N-2s}{2}}, \text{ where } \mu_{N,s} = c_{N,s}^{\frac{4}{N-2s}}.$$

Proof. Using Lemma 3.2, we obtain $\Omega_\varepsilon \mapsto \mathbb{R}^N$ as $\varepsilon \rightarrow 0$. We know z_ε satisfies Eq.(3.9). Note that, $\max_\Omega u_\varepsilon(x) = u_\varepsilon(x_\varepsilon)$ implies z_ε attains maximum at 0 and $z_\varepsilon(0) = 1$. Therefore, applying the definition of fractional laplace operator, it is easy to see that $(-\Delta)^s z_\varepsilon(0) \geq 0$. Thus from (3.9), we have $1 - \varepsilon \gamma_\varepsilon^{\frac{(N+2s)-q(N-2s)}{2}} \geq 0$. This in turn implies, $\lim_{\varepsilon \rightarrow 0} \varepsilon \gamma_\varepsilon^{\frac{(N+2s)-q(N-2s)}{2}} \in [0, 1]$. Consequently, using Schauder estimate [27], $z_\varepsilon \rightarrow Z$ in $C_{loc}^{2s-\delta}(\mathbb{R}^N)$, for some $\delta > 0$. Let $\phi \in C_0^\infty(\mathbb{R}^N)$. Thus, $\phi \in C_0^\infty(\Omega_\varepsilon)$ for ε small. Taking ϕ as the test function, from Eq.(3.9) we have

$$(3.10) \quad \begin{aligned} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(z_\varepsilon(x) - z_\varepsilon(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} dx dy &= \int_{\Omega_\varepsilon} z_\varepsilon^{2^*-1} \phi dx \\ &- \varepsilon \gamma_\varepsilon^{\frac{(N+2s)-q(N-2s)}{2}} \int_{\Omega_\varepsilon} z_\varepsilon^q \phi dx. \end{aligned}$$

As $\|z_\varepsilon\|_{L^\infty} = 1$ and ϕ has compact support, using dominated convergence theorem as in the proof of Lemma 3.1, we can pass to the limit $\varepsilon \rightarrow 0$ in the above integral

identity and obtain

$$(3.11) \quad \begin{cases} (-\Delta)^s Z = Z^{2^*-1} - cZ^q & \text{in } \mathbb{R}^N, \\ 0 < Z \leq 1, & \text{in } \mathbb{R}^N, \quad Z(0) = 1, \end{cases}$$

where $c = \lim_{\varepsilon \rightarrow 0} \varepsilon \gamma_\varepsilon^{\frac{(N+2s)-q(N-2s)}{2}}$. Since $z_\varepsilon \in H^s(\Omega_\varepsilon)$ and $z_\varepsilon = 0$ in $\mathbb{R}^N \setminus \Omega_\varepsilon$, multiplying (3.9) by z_ε and integrating over \mathbb{R}^N , we have

$$\|z_\varepsilon\|_{H^s(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} z_\varepsilon^{2^*} dx - \varepsilon \gamma_\varepsilon^{\frac{(N+2s)-q(N-2s)}{2}} \int_{\mathbb{R}^N} z_\varepsilon^{q+1} dx \leq \int_{\Omega_\varepsilon} z_\varepsilon^{2^*} dx < B.$$

Therefore, up to a subsequence $z_\varepsilon \rightharpoonup \tilde{Z}$ in $H^s(\mathbb{R}^N)$. By the uniqueness of limit, $Z = \tilde{Z}$. Thus $Z \in H^s(\mathbb{R}^N)$. Consequently, multiplying (3.11) by Z and integrating over \mathbb{R}^N , we get $Z \in L^{q+1}(\mathbb{R}^N)$. Hence, if $c \neq 0$, we get a contradiction by Pohozaev identity (see [4, Theorem 1.4]). This implies $c = 0$ and Z satisfies (1.13). As a consequence, Z must be of the form $\xi^{-\frac{N-2s}{2}} U(\frac{x}{\xi})$, for some $\xi > 0$, where U is as in (1.11). As, $\max_{\Omega_\varepsilon} z_\varepsilon = z_\varepsilon(0) = 1$, we get $Z(0) = 1$ and $0 \leq Z \leq 1$. Using this fact, it is easy to see that $\xi = c_{N,s}^{\frac{2}{N-2s}}$, where $c_{N,s}$ is as defined in (1.12). From this, a computation yields $Z(x) = \left[1 + \frac{|x|^2}{\mu_{N,s}}\right]^{-\frac{N-2s}{2}}$, where $\mu_{N,s} = c_{N,s}^{\frac{4}{N-2s}}$. \square

Now we show that there exists $C > 0$ independent of $\varepsilon > 0$ such that

$$(3.12) \quad z_\varepsilon(x) \leq CZ(x) \text{ for all } x \in \Omega_\varepsilon.$$

The local behavior of z_ε is known. Next, we need to check the behavior of z_ε near ∞ . For this, define the Kelvin transform of z_ε as

$$(3.13) \quad \hat{z}_\varepsilon(x) = |x|^{-(N-2s)} z_\varepsilon\left(\frac{x}{|x|^2}\right) \text{ in } \Omega_\varepsilon \setminus \{0\}.$$

From (3.9), it follows that \hat{z}_ε satisfies

$$(3.14) \quad \begin{cases} (-\Delta)^s \hat{z}_\varepsilon = \hat{z}_\varepsilon^{2^*-1} - \varepsilon \gamma_\varepsilon^{\frac{(N+2s)-q(N-2s)}{2}} |x|^{q(N-2s)-(N+2s)} \hat{z}_\varepsilon^q & \text{in } \Omega_\varepsilon^* \\ \hat{z}_\varepsilon = 0 & \text{in } \mathbb{R}^N \setminus \Omega_\varepsilon^*. \end{cases}$$

where Ω_ε^* is the image Ω_ε under the Kelvin transform. Hence the behavior of z_ε near ∞ amounts to study the behavior of \hat{z}_ε near 0.

Lemma 3.4. *There exist $R > 0$ and $C > 0$ independent of $\varepsilon > 0$ such that any solution of (3.14) satisfy*

$$(3.15) \quad \|\hat{z}_\varepsilon\|_{L^\infty(B_R)} \leq C \left(\int_{B_R} \hat{z}_\varepsilon^{2^*} dx \right)^{\frac{1}{2^*}}.$$

Proof. The proof follows along the same line of arguments as in [4, Theorem 1.1] (see also [33]) with a suitable modification and we skip the proof. \square

For (3.12), note that $\|z_\varepsilon\|_\infty = 1$ and this implies that $z_\varepsilon \leq CZ(x)$ locally. From (1.22) and (3.8), it follows

$$A < \int_{\Omega_\varepsilon} z_\varepsilon^{2^*} dx < B.$$

But this implies that

$$\int_{B_R \cap \Omega_\varepsilon^*} \hat{z}_\varepsilon^{2^*} dx \leq \int_{\Omega_\varepsilon^*} \hat{z}_\varepsilon^{2^*} dx = \int_{\Omega_\varepsilon} z_\varepsilon^{2^*} dx < B.$$

Consequently from Lemma 3.4, we obtain $z_\varepsilon(x) \leq \frac{C}{|x|^{N-2s}}$ as $|x| \rightarrow \infty$. Moreover, since at infinity Z decays as $|x|^{-(N-2s)}$, we conclude $z_\varepsilon \leq CZ(x)$ near infinity. Hence, we have $z_\varepsilon \leq CZ(x)$ for all $x \in \Omega_\varepsilon$. As a conclusion, from (3.8) we obtain that there exists $C > 0$ independent of ε such that

$$(3.16) \quad u_\varepsilon(x) \leq C\gamma_\varepsilon^{-\frac{N-2s}{2}} Z\left(\frac{x-x_\varepsilon}{\gamma_\varepsilon}\right).$$

Define $w_\varepsilon(x) = \|u_\varepsilon\|_\infty u_\varepsilon(x) = \gamma_\varepsilon^{-\frac{N-2s}{2}} u_\varepsilon(x)$. Then w_ε satisfies

$$(3.17) \quad \begin{cases} (-\Delta)^s w_\varepsilon = \gamma_\varepsilon^{-\frac{N-2s}{2}} u_\varepsilon^{2^*-1} - \varepsilon \gamma_\varepsilon^{-\frac{N-2s}{2}} u_\varepsilon^q & \text{in } \Omega \\ w_\varepsilon = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Lemma 3.5. *The Green function associated to the fractional laplacian $(-\Delta)^s$ satisfy the following inequalities.*

- (i) $G(x, y) \leq \frac{C}{|x-y|^{N-2s}}$ and
- (ii) $G(x, y) \leq \frac{Cd^s(x)}{|x-y|^{N-s}}$.

where $C > 0$ is a constant depending on Ω and s and $N > 2s$.

Proof. This follows from Chen and Song [9, Theorem 1.1]. \square

Lemma 3.6. *Let w_ε be as in (3.17). Then there exists a constant $C > 0$ such that*

$$\|w_\varepsilon\|_{L^\infty(K)} \leq C,$$

for every $K \subset \subset \Omega \setminus \{x_0\}$.

Proof. From the Green function representation and Lemma 3.5 we have

$$\begin{aligned} |w_\varepsilon(x)| &\leq \gamma_\varepsilon^{-\frac{N-2s}{2}} \int_{\Omega} G(x, y) u_\varepsilon^{2^*-1} dy + \varepsilon \gamma_\varepsilon^{-\frac{N-2s}{2}} \int_{\Omega} G(x, y) u_\varepsilon^q dy \\ &\leq C\gamma_\varepsilon^{-\frac{N-2s}{2}} \int_{\Omega} |x-y|^{2s-N} u_\varepsilon^{2^*-1} dy + C\varepsilon \gamma_\varepsilon^{-\frac{N-2s}{2}} \int_{\Omega} |x-y|^{2s-N} u_\varepsilon^q dy. \end{aligned}$$

Moreover,

$$\begin{aligned} \gamma_\varepsilon^{-\frac{N-2s}{2}} \int_{\Omega} |x-y|^{2s-N} u_\varepsilon^{2^*-1} dy &= \gamma_\varepsilon^{-\frac{N-2s}{2}} \int_{\Omega \cap B_{\frac{|x-x_\varepsilon|}{2}}(x_\varepsilon)} |x-y|^{2s-N} u_\varepsilon^{2^*-1} dy \\ &\quad + \gamma_\varepsilon^{-\frac{N-2s}{2}} \int_{\Omega \setminus \Omega \cap B_{\frac{|x-x_\varepsilon|}{2}}(x_\varepsilon)} |x-y|^{2s-N} u_\varepsilon^{2^*-1} dy. \end{aligned}$$

Using (3.16) along with that fact that $Z(x) = |x|^{-(N-2s)}$ at infinity, we have

$$\gamma_\varepsilon^{-\frac{N-2s}{2}} |x-y|^{2s-N} u_\varepsilon^{2^*-1}(y) \leq \frac{C\gamma_\varepsilon^{2s}}{|x-y|^{N-2s}|y-x_\varepsilon|^{N+2s}} \quad \text{if } y \in \Omega \setminus B_{\frac{|x-x_\varepsilon|}{2}}(x_\varepsilon)$$

and

$$\varepsilon \gamma_\varepsilon^{-\frac{N-2s}{2}} |x-y|^{2s-N} u_\varepsilon^q(y) dy \leq \frac{C\varepsilon \gamma_\varepsilon^{\frac{(N-2s)(q-1)}{2}}}{|x-y|^{N-2s}|y-x_\varepsilon|^{(N-2s)q}} \quad \text{if } y \in \Omega \setminus B_{\frac{|x-x_\varepsilon|}{2}}(x_\varepsilon).$$

Hence,

$$\begin{aligned}
& \gamma_\varepsilon^{-\frac{N-2s}{2}} \int_{\Omega \setminus B_{\frac{|x-x_\varepsilon|}{2}}(x_\varepsilon)} |x-y|^{2s-N} u_\varepsilon^{2^*-1}(y) dy \\
& \leq \frac{C}{|x-x_\varepsilon|^{N+2s}} \int_{\Omega \setminus B_{\frac{|x-x_\varepsilon|}{2}}(x_\varepsilon)} \frac{1}{|x-y|^{N-2s}} dy \\
& \leq \frac{C}{|x-x_\varepsilon|^{N+2s}}
\end{aligned}$$

and

$$\begin{aligned}
& \varepsilon \gamma_\varepsilon^{-\frac{N-2s}{2}} \int_{\Omega \setminus \Omega \cap B_{\frac{|x-x_\varepsilon|}{2}}(x_\varepsilon)} |x-y|^{2s-N} u_\varepsilon^q(y) dy \\
& \leq \frac{C \varepsilon \gamma_\varepsilon^{\frac{(N-2s)(q-1)}{2}}}{|x-x_\varepsilon|^{(N-2s)q}} \int_{\Omega \setminus B_{\frac{|x-x_\varepsilon|}{2}}(x_\varepsilon)} \frac{1}{|x-y|^{N-2s}} dy \\
& \leq \frac{C}{|x-x_\varepsilon|^{(N-2s)q}}.
\end{aligned}$$

When $y \in \Omega \cap B_{\frac{|x-x_\varepsilon|}{2}}(x_\varepsilon)$, we have $|x-y| \geq |x-x_\varepsilon| - |y-x_\varepsilon| \geq \frac{1}{2}|x-x_\varepsilon|$. Therefore applying (3.16) we obtain

$$\begin{aligned}
\gamma_\varepsilon^{-\frac{N-2s}{2}} \int_{\Omega \cap B_{\frac{|x-x_\varepsilon|}{2}}(x_\varepsilon)} |x-y|^{2s-N} u_\varepsilon^{2^*-1}(y) dy & \leq \frac{C \gamma_\varepsilon^{-\frac{N-2s}{2}}}{|x-x_\varepsilon|^{N-2s}} \int_{\Omega \cap B_{\frac{|x-x_\varepsilon|}{2}}(x_\varepsilon)} u_\varepsilon^{2^*-1}(y) dy \\
& \leq \frac{C \gamma_\varepsilon^{-N}}{|x-x_\varepsilon|^{N-2s}} \int_{\mathbb{R}^N} Z^{2^*-1}\left(\frac{y-x_\varepsilon}{\gamma_\varepsilon}\right) dy \\
& \leq \frac{C}{|x-x_\varepsilon|^{N-2s}} \int_{\mathbb{R}^N} Z^{2^*-1}(x) dx \\
& \leq \frac{C}{|x-x_\varepsilon|^{N-2s}}.
\end{aligned}$$

Similarly applying Lemma 3.3, we obtain

$$\begin{aligned}
\varepsilon \gamma_\varepsilon^{-\frac{N-2s}{2}} \int_{\Omega \cap B_{\frac{|x-x_\varepsilon|}{2}}(x_\varepsilon)} |x-y|^{2s-N} u_\varepsilon^q(y) dy & \leq \frac{C \varepsilon \gamma_\varepsilon^{-\frac{N-2s}{2}}}{|x-x_\varepsilon|^{N-2s}} \int_{\Omega \cap B_{\frac{|x-x_\varepsilon|}{2}}(x_\varepsilon)} u_\varepsilon^q(y) dy \\
& \leq \frac{C \varepsilon \gamma_\varepsilon^{N-\frac{N-2s}{2}(q+1)}}{|x-x_\varepsilon|^{N-2s}} \int_{\mathbb{R}^N} Z^q(y) dy \\
& \leq \frac{C}{|x-x_\varepsilon|^{N-2s}}.
\end{aligned}$$

where $C > 0$ is a uniform constant. Hence for any compact set $K \subset \Omega \setminus \{x_0\}$, we have $K \subset \subset \Omega \setminus \{x_\varepsilon\}$, for $\varepsilon > 0$ small enough and therefore, we have $\|w_\varepsilon\|_{L^\infty(K)} \leq C$. \square

Note that (3.17) can be rewritten as

$$(3.18) \quad \begin{cases} (-\Delta)^s w_\varepsilon = \gamma_\varepsilon^{2s} w_\varepsilon^{2^*-1} - \varepsilon \gamma_\varepsilon^{\frac{N-2s}{2}(q-1)} w_\varepsilon^q & \text{in } \Omega \\ w_\varepsilon = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Lemma 3.7.

$$(3.19) \quad \lim_{\varepsilon \rightarrow 0} \frac{w_\varepsilon(x)}{d(x)^s} = \gamma_0 \frac{G(x, x_0)}{d(x)^s} \text{ in } C(\bar{\Omega} \setminus B_r(x_0)),$$

for any $r > 0$. Here, γ_0 is same as in Lemma 3.8.

Proof. Choose $r > 0$ such that $\Omega' = \Omega \setminus \bar{B}_r(x_0)$ is connected. Thus by Lemma 3.6, $|w_\varepsilon| \leq C$ for all $x \in \Omega'$. Let $x \in \bar{\Omega} \setminus B_r(x_0)$,

Then for any $r > 0$ small and the fact that $\gamma_\varepsilon \rightarrow 0$ we have

$$\begin{aligned} \frac{w_\varepsilon(x)}{d(x)^s} &= \frac{\gamma_\varepsilon^{-\frac{N-2s}{2}}}{d(x)^s} \int_{\Omega} G(x, y) u_\varepsilon^{2^*-1} dy - \varepsilon \frac{\gamma_\varepsilon^{-\frac{N-2s}{2}}}{d(x)^s} \int_{\Omega} G(x, y) u_\varepsilon^q dy \\ &= \frac{\gamma_\varepsilon^{-\frac{N-2s}{2}}}{d(x)^s} \int_{B_r(x_0)} G(x, y) u_\varepsilon^{2^*-1}(y) dy + \frac{\gamma_\varepsilon^{-\frac{N-2s}{2}}}{d(x)^s} \int_{\Omega \setminus B_r(x_0)} G(x, y) u_\varepsilon^{2^*-1}(y) dy \\ (3.20) \quad &- \varepsilon \frac{\gamma_\varepsilon^{-\frac{N-2s}{2}}}{d(x)^s} \int_{B_r(x_0)} G(x, y) u_\varepsilon^q dy - \varepsilon \frac{\gamma_\varepsilon^{-\frac{N-2s}{2}}}{d(x)^s} \int_{\Omega \setminus B_r(x_0)} G(x, y) u_\varepsilon^q dy. \end{aligned}$$

Using the second estimate in Lemma 3.5, (3.16) and the fact that Z decays at infinity of the order $|y|^{-(N-2s)}$, we estimate the 2nd term on RHS as follows

$$\begin{aligned} \frac{\gamma_\varepsilon^{-\frac{N-2s}{2}}}{d(x)^s} \int_{\Omega \setminus B_r(x_0)} G(x, y) u_\varepsilon^{2^*-1}(y) dy &\leq \gamma_\varepsilon^{-\frac{N-2s}{2}} \int_{\Omega \setminus B_r(x_0)} \frac{u_\varepsilon^{2^*-1}(y)}{|x-y|^{N-s}} dy \\ &= \gamma_\varepsilon^{-N} \int_{\Omega \setminus B_r(x_0)} Z^{2^*-1} \left(\frac{y-x_\varepsilon}{\gamma_\varepsilon} \right) |x-y|^{s-N} dy \\ &\leq C \gamma_\varepsilon^{-N} \int_{\Omega \setminus B_r(x_0)} \left| \frac{y-x_\varepsilon}{\gamma_\varepsilon} \right|^{-(N+2s)} \frac{1}{|x-y|^{N-s}} dy \\ &= C \gamma_\varepsilon^{2s} \int_{\Omega \setminus B_r(x_0)} \frac{1}{|y-x_\varepsilon|^{(N+2s)} |x-y|^{N-s}} dy \\ &= o_{r,\varepsilon}(1), \end{aligned}$$

where $o_{r,\varepsilon}(1)$ denote the term going to 0 as $r \rightarrow 0$ or $\varepsilon \rightarrow 0$. Note that we have used the fact that $|x-y|^{s-N}$ is integrable in Ω . Similarly, it can be shown that,

$$\begin{aligned} \frac{\gamma_\varepsilon^{-\frac{N-2s}{2}}}{d(x)^s} \int_{\Omega \setminus B_r(x_0)} G(x, y) u_\varepsilon^q(y) dy &\leq \gamma_\varepsilon^{-\frac{N-2s}{2}} \int_{\Omega \setminus B_r(x_0)} \frac{u_\varepsilon^q(y)}{|x-y|^{N-s}} dy \\ &\leq C \gamma_\varepsilon^{(\frac{N-2s}{2})(q-1)} \int_{\Omega \setminus B_r(x_0)} \frac{1}{|y-x_\varepsilon|^{(N-2s)q} |x-y|^{N-s}} dy \\ &= o_{r,\varepsilon}(1). \end{aligned}$$

Furthermore $\frac{G(x, \cdot)}{d(x)^s}$ is continuous in $\bar{\Omega} \setminus \{x\}$, (see [9, Lemma 6.5]). Therefore, from (3.20) we obtain

$$(3.21) \quad \frac{w_\varepsilon(x)}{d(x)^s} = \gamma_\varepsilon^{-\frac{N-2s}{2}} \frac{G(x, x_0)}{d(x)^s} \int_{B_r(x_0)} u_\varepsilon^{2^*-1} dy + L + o_{\varepsilon,r}(1),$$

where

$$L = \varepsilon \gamma_\varepsilon^{-\frac{N-2s}{2}} \frac{G(x, x_0)}{d(x)^s} \int_{B_r(x_0)} u_\varepsilon^q dy.$$

Doing a straight forward computation using (3.16), we have

$$\begin{aligned} L &\leq \varepsilon \gamma_\varepsilon \frac{(N+2s)-q(N-2s)}{2} \frac{G(x, x_0)}{d(x)^s} \int_{\frac{B_r(x_0)-x_0}{\gamma_\varepsilon}} Z^q(y) dy \\ &\leq \varepsilon \gamma_\varepsilon \frac{(N+2s)-q(N-2s)}{2} \frac{G(x, x_0)}{d(x)^s} \int_{\mathbb{R}^N} Z^q(y) dy \end{aligned}$$

Thus, using Lemma 3.3, it is not difficult to check that $L = o_{\varepsilon, r}(1)$. Define

$$\gamma_0 = \lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \gamma_\varepsilon^{-\frac{N-2s}{2}} \int_{B_r(0)} u_\varepsilon^{2^*-1} dy.$$

Then it follows from (3.21) that

$$\lim_{\varepsilon \rightarrow 0} \frac{w_\varepsilon(x)}{d(x)^s} = \gamma_0 \frac{G(x, x_0)}{d(x)^s}.$$

This argument actually goes through for uniform convergence. Clearly, γ_0 is positive as

$$\begin{aligned} \gamma_0 &\geq \lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \gamma_\varepsilon^{-\frac{N-2s}{2}} \|u_\varepsilon\|_\infty^{-1} \int_{B_r(0)} u_\varepsilon^{2^*} dy \\ &\geq \lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_{B_r(0)} u_\varepsilon^{2^*} dy \\ &\geq A. \end{aligned}$$

This completes the proof. \square

Lemma 3.8. *Let u_ε be as in Theorem 1.2 and γ_ε be as defined in (3.5). Define*

$$\gamma_0 := \lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \gamma_\varepsilon^{-\frac{N-2s}{2}} \int_{B_r(x_0)} u_\varepsilon^{2^*-1} dy. \text{ Then}$$

$$(3.22) \quad \gamma_0 = \frac{\omega_N c_{N,s}^{2^*}}{2} \frac{\Gamma(\frac{N}{2})\Gamma(s)}{\Gamma(\frac{N+2s}{2})},$$

where $c_{N,s}$ is as defined in (1.12).

Proof. We define $I_{\varepsilon, r} := \gamma_\varepsilon^{-\frac{N-2s}{2}} \int_{B_r(x_0)} u_\varepsilon^{2^*-1} dy$. Using (3.8), we obtain $u_\varepsilon(x) = \gamma_\varepsilon^{-\frac{N-2s}{2}} z_\varepsilon\left(\frac{x-x_\varepsilon}{\gamma_\varepsilon}\right)$. Thus

$$(3.23) \quad I_{\varepsilon, r} = \gamma_\varepsilon^{-\frac{N-2s}{2} - \frac{N+2s}{2} + N} \int_{\frac{B_r(x_0)-x_\varepsilon}{\gamma_\varepsilon}} z_\varepsilon^{2^*-1}(x) dx = \int_{\frac{B_r(x_0)-x_\varepsilon}{\gamma_\varepsilon}} z_\varepsilon^{2^*-1}(x) dx.$$

Note that, $\varepsilon \rightarrow 0$ implies $\gamma_\varepsilon \rightarrow 0$. Therefore,

$$(3.24) \quad \gamma_0 = \lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} I_{\varepsilon, r} = \int_{\mathbb{R}^N} Z^{2^*-1} dx,$$

where Z is as in Lemma 3.3. Hence, by doing a straight forward computation, we obtain

$$\gamma_0 = \frac{\omega_N c_{N,s}^{2^*}}{2} B\left(\frac{N}{2}, s\right),$$

where $B(a, b) = \int_0^\infty t^{a-1}(1+t)^{-a-b}dt$ is the Beta function, $c_{N,s}$ is as defined in (1.12) and ω_N is the surface measure of unit ball. Recall that $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$. Thus $B(\frac{N}{2}, s) = \frac{\Gamma(\frac{N}{2})\Gamma(s)}{\Gamma(\frac{N+2s}{2})}$ and the lemma follows. \square

Proof of Theorem 1.2. Applying (1.21) to u_ε yields

$$\Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{u_\varepsilon(x)}{d^s(x)} \right)^2 \langle x - x_0, \nu \rangle dS = 2\varepsilon \left(\frac{N-2s}{2} - \frac{N}{q+1} \right) \int_{\Omega} u_\varepsilon^{q+1} dx.$$

Using $w_\varepsilon = \|u_\varepsilon\|_\infty u_\varepsilon$ in the above expression, we have
(3.25)

$$\Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{w_\varepsilon(x)}{d^s(x)} \right)^2 \langle x - x_0, \nu \rangle dS = 2\varepsilon \left(\frac{N-2s}{2} - \frac{N}{q+1} \right) \|u_\varepsilon\|_\infty^2 \int_{\Omega} u_\varepsilon^{q+1} dx.$$

Thanks to Lemma 3.7, applying dominated convergence theorem, we have
(3.26)

$$\lim_{\varepsilon \rightarrow 0} \Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{w_\varepsilon(x)}{d^s(x)} \right)^2 \langle x - x_0, \nu \rangle dS = \gamma_0^2 \Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{G(x, x_0)}{d^s(x)} \right)^2 \langle x - x_0, \nu \rangle dS.$$

Moreover, using the relations (3.8) and (3.5), the RHS of (3.25) reduces to

$$\begin{aligned} \text{RHS of (3.25)} &= 2\varepsilon \left(\frac{N-2s}{2} - \frac{N}{q+1} \right) \|u_\varepsilon\|_\infty^2 \gamma_\varepsilon^{\frac{(N+2s)-q(N-2s)}{2}} \int_{\Omega_\varepsilon} z_\varepsilon^{q+1} dx \\ (3.27) \quad &= 2\varepsilon \left(\frac{N-2s}{2} - \frac{N}{q+1} \right) \|u_\varepsilon\|_\infty^{\frac{q(N-2s)+N-6s}{N-2s}} \int_{\Omega_\varepsilon} z_\varepsilon^{q+1} dx. \end{aligned}$$

Since $z_\varepsilon \rightarrow Z$ a.e and $z_\varepsilon \leq CZ$, by the dominated convergence theorem it follows $\int_{\Omega_\varepsilon} z_\varepsilon^{q+1} dx \rightarrow \int_{\mathbb{R}^N} Z^{q+1} dx$. We substitute back (3.27) into (3.25) and take the limit $\varepsilon \rightarrow 0$. Therefore, using (3.26) we obtain

$$(3.28) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon \|u_\varepsilon\|_\infty^{\frac{q(N-2s)+N-6s}{N-2s}} = \frac{\gamma_0^2 \Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{G(x, x_0)}{d^s(x)} \right)^2 \langle x - x_0, \nu \rangle dS}{2 \left(\frac{N-2s}{2} - \frac{N}{q+1} \right) \int_{\mathbb{R}^N} Z^{q+1} dx}.$$

From Lemma 3.3, we know $Z(x) = \left(1 + \frac{|x|^2}{\mu_{N,s}}\right)^{-\left(\frac{N-2s}{2}\right)}$, where $\mu_{N,s} = c_{N,s}^{\frac{4}{N-2s}}$. Thus, a straight forward calculation yields

$$\int_{\mathbb{R}^N} Z^{q+1} dx = \frac{c_{N,s}^{2*} \omega_N}{2} B\left(\frac{N}{2}, \left(\frac{N-2s}{2}\right)q - s\right).$$

From Lemma 3.8, it is known that $\gamma_0 = \frac{\omega_N c_{N,s}^{2^*}}{2} \frac{\Gamma(\frac{N}{2})\Gamma(s)}{\Gamma(\frac{N+2s}{2})}$. Substituting the value of γ_0 and $\int_{\mathbb{R}^N} Z^{q+1} dx$ in (3.28) we have,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \|u_\varepsilon\|_\infty^{\frac{q(N-2s)+N-6s}{N-2s}} = \frac{\omega_N c_{N,s}^{2^*}}{2} \frac{(q+1)R_{N,s,x_0}}{q(N-2s) - (N+2s)} s^2 \Gamma(s)^2 B\left(\frac{N}{2}, s\right)^2 \times \\ B\left(\frac{N}{2}, \left(\frac{N-2s}{2}\right)q - s\right)^{-1}$$

□

4. Uniqueness result for $p = 2^* - 1$

Proof of Theorem 1.3: We break the proof into few steps.

Step 1: Let u_ε and v_ε be two solutions of (1.2) with

$$\max_{\Omega} u_\varepsilon = \max_{\Omega} v_\varepsilon.$$

Let γ_ε be as in (3.5). Then by the assumptions of the theorem, we have

$$\gamma_\varepsilon = \|u_\varepsilon\|_{L^\infty}^{-\frac{2}{N-2s}} = u_\varepsilon(0)^{-\frac{2}{N-2s}} = \|v_\varepsilon\|_{L^\infty(\Omega)}^{-\frac{2}{N-2s}} = v_\varepsilon(0)^{-\frac{2}{N-2s}}.$$

Note that, by Lemma 3.1, we have $\gamma_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Define,

$$\theta_\varepsilon(x) = u_\varepsilon(\gamma_\varepsilon x) - v_\varepsilon(\gamma_\varepsilon x), \quad x \in \Omega_\varepsilon = \frac{\Omega}{\gamma_\varepsilon},$$

and

$$\psi_\varepsilon(x) = \frac{\theta_\varepsilon(x)}{\|\theta_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}} = \frac{\theta_\varepsilon(x)}{\|u_\varepsilon - v_\varepsilon\|_{L^\infty(\Omega)}}.$$

Therefore,

$$(-\Delta)^s \psi_\varepsilon = \frac{\gamma_\varepsilon^{2s}}{\|u_\varepsilon - v_\varepsilon\|_{L^\infty(\Omega)}} [(u_\varepsilon^p(\gamma_\varepsilon x) - v_\varepsilon^p(\gamma_\varepsilon x)) - \varepsilon (u_\varepsilon^q(\gamma_\varepsilon x) - v_\varepsilon^q(\gamma_\varepsilon x))].$$

It is easy to see that,

$$u_\varepsilon^p(\gamma_\varepsilon x) - v_\varepsilon^p(\gamma_\varepsilon x) = p \int_0^1 (tu_\varepsilon(\gamma_\varepsilon x) + (1-t)v_\varepsilon(\gamma_\varepsilon x))^{p-1} \theta_\varepsilon(x) dt.$$

Using the fact that $p = 2^* - 1 = \frac{N+2s}{N-2s}$ and $\gamma_\varepsilon^{2s} = \|u_\varepsilon\|_{L^\infty(\Omega)}^{-(p-1)} = \|v_\varepsilon\|_{L^\infty(\Omega)}^{-(p-1)}$, a straight forward computation yields

$$(4.1) \quad \begin{cases} (-\Delta)^s \psi_\varepsilon = (c_\varepsilon^1(x) - \varepsilon c_\varepsilon^2(x)) \psi_\varepsilon & \text{in } \Omega_\varepsilon, \\ \psi_\varepsilon = 0 & \text{in } \mathbb{R}^N \setminus \Omega_\varepsilon, \end{cases}$$

where

$$(4.2) \quad c_\varepsilon^1(x) = p \int_0^1 \left[t \frac{u_\varepsilon(\gamma_\varepsilon x)}{\|u_\varepsilon\|_{L^\infty(\Omega)}} + (1-t) \frac{v_\varepsilon(\gamma_\varepsilon x)}{\|v_\varepsilon\|_{L^\infty(\Omega)}} \right]^{p-1} dt,$$

$$(4.3) \quad c_\varepsilon^2(x) = q \gamma_\varepsilon^{\frac{(N+2s)-q(N-2s)}{2}} \int_0^1 \left[t \frac{u_\varepsilon(\gamma_\varepsilon x)}{\|u_\varepsilon\|_{L^\infty(\Omega)}} + (1-t) \frac{v_\varepsilon(\gamma_\varepsilon x)}{\|v_\varepsilon\|_{L^\infty(\Omega)}} \right]^{q-1} dt.$$

Here we observe that, $\frac{u_\varepsilon(\gamma_\varepsilon x)}{\|u_\varepsilon\|_{L^\infty(\Omega)}} = z_\varepsilon(x)$, where z_ε is as defined in (3.8) (since here $x_\varepsilon = 0$). Consequently, using Lemma 3.3 and (3.12), we obtain

$$(4.4) \quad \frac{u_\varepsilon(\gamma_\varepsilon x)}{\|u_\varepsilon\|_{L^\infty(\Omega)}} \rightarrow Z \quad \text{in } C_{loc}^s(\mathbb{R}^N) \quad \text{and} \quad \frac{u_\varepsilon(\gamma_\varepsilon x)}{\|u_\varepsilon\|_{L^\infty(\Omega)}} \leq \frac{C}{\left(1 + \frac{|x|^2}{\mu_{N,s}}\right)^{\frac{N-2s}{2}}},$$

where Z is the solution of (1.13) with $Z(0) = 1$ and $0 < Z \leq 1$. Hence $Z(x) = \left(1 + \frac{|x|^2}{\mu_{N,s}}\right)^{-\frac{N-2s}{2}}$, where $\mu_{N,s} = c_{N,s}^{\frac{4}{N-2s}}$, (see Lemma 3.3). As a consequence, thanks to Lemma 3.3(i), from (4.2) and (4.3) we have

$$(4.5) \quad c_\varepsilon^1(x) \rightarrow \left(\frac{N+2s}{N-2s}\right) \frac{1}{\left(1 + \frac{|x|^2}{\mu_{N,s}}\right)^{2s}} \quad \text{and} \quad \varepsilon c_\varepsilon^2(x) \rightarrow 0,$$

uniformly on compact subsets of \mathbb{R}^N . Applying Schauder estimates [27] to the equation (4.1), it follows there exists $\psi \in C^s(\mathbb{R}^N)$ such that $\psi_\varepsilon \rightarrow \psi$ in $C_{loc}^s(\mathbb{R}^N)$. Since, from Remark 1.2 we have ψ_ε is radially symmetric, we obtain ψ is radially symmetric too. Passing to the limit in (4.1) (as in Lemma 3.3) yields

$$(4.6) \quad \begin{cases} (-\Delta)^s \psi = \left(\frac{N+2s}{N-2s}\right) \frac{\psi}{\left(1 + \frac{|x|^2}{\mu_{N,s}}\right)^{2s}} & \text{in } \mathbb{R}^N, \\ \|\psi\|_{L^\infty(\mathbb{R}^N)} \leq 1. \end{cases}$$

Step 2: In this step, we will prove that $\psi \in H^s(\mathbb{R}^N)$.

Since $\psi_\varepsilon \in H^s(\Omega_\varepsilon)$, $\psi_\varepsilon = 0$ in $\mathbb{R}^N \setminus \Omega_\varepsilon$ and $u_\varepsilon, v_\varepsilon = 0$ in $\mathbb{R}^N \setminus \Omega$, taking ψ_ε as a test function in (4.1), we have

$$(4.7) \quad \|\psi_\varepsilon\|_{H^s(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} c_\varepsilon^1(x) \psi_\varepsilon^2 dx - \varepsilon \int_{\mathbb{R}^N} c_\varepsilon^2(x) \psi_\varepsilon^2 dx \leq \int_{\Omega_\varepsilon} c_\varepsilon^1(x) \psi_\varepsilon^2 dx.$$

Thus applying the Sobolev inequality, we have

$$(4.8) \quad \mathcal{S} \left(\int_{\Omega_\varepsilon} |\psi_\varepsilon|^{2^*} dx \right)^{\frac{2}{2^*}} \leq \int_{\Omega_\varepsilon} c_\varepsilon^1(x) \psi_\varepsilon^2 dx.$$

Let us fix $\delta > 0$, will be chosen later. Since $\|\psi_\varepsilon\|_{L^\infty(\Omega)} = 1$, Hölder inequality yields

$$(4.9) \quad \int_{\Omega_\varepsilon} c_\varepsilon^1(x) \psi_\varepsilon^2 dx \leq \int_{\Omega_\varepsilon} c_\varepsilon^1(x) \psi_\varepsilon^{2-\delta} dx \leq \left(\int_{\Omega_\varepsilon} |\psi_\varepsilon|^{2^*} dx \right)^{\frac{2-\delta}{2^*}} \left(\int_{\Omega_\varepsilon} |c_\varepsilon^1|^{\frac{2^*}{2^*-2+\delta}} dx \right)^{\frac{2^*-2+\delta}{2^*}}.$$

Combining (4.8) and (4.9) we have

$$(4.10) \quad \begin{aligned} \int_{\Omega_\varepsilon} |\psi_\varepsilon|^{2^*} dx &\leq \left(\int_{\Omega_\varepsilon} |c_\varepsilon^1|^{\frac{2^*}{2^*-2+\delta}} dx \right)^{\frac{2^*-2+\delta}{\delta}} \\ &\leq C \left(\int_{\mathbb{R}^N} \left[\frac{1}{\left(1 + \frac{|x|^2}{\mu_{N,s}}\right)^{2s}} \right]^{\frac{2^*}{2^*-2+\delta}} dx \right)^{\frac{2^*-2+\delta}{\delta}} \\ &\leq C, \end{aligned}$$

for some constant $C > 0$, if we choose $\delta < \frac{4s}{N-2s}$. For this choice of δ , substituting back (4.10) into (4.9) yields $\int_{\Omega_\varepsilon} c_\varepsilon^1(x) \psi_\varepsilon^2 dx \leq C$. As a result, from (4.7) we have

$\|\psi_\varepsilon\|_{H^s(\mathbb{R}^N)}$ is uniformly bounded. Since $\psi_\varepsilon \rightarrow \psi$ in $C_{loc}^s(\mathbb{R}^N)$,

$$\|\psi\|_{H^s(\mathbb{R}^N)} \leq \liminf_{\varepsilon \rightarrow 0} \|\psi_\varepsilon\|_{H^s(\mathbb{R}^N)} \leq C,$$

which implies $\psi \in H^s(\mathbb{R}^N)$.

Step 3: In this step we will establish that

$$(4.11) \quad |\psi_\varepsilon(x)| \leq \frac{C}{|x|^{N-2s}}, \quad x \in \Omega_\varepsilon \setminus B_r(0),$$

for $\varepsilon > 0$ small enough and for some constant $C > 0$ and $r > 0$ independent of ε .

To prove this step, define $\hat{\psi}_\varepsilon$ as the Kelvin transform of ψ_ε , that is,

$$\hat{\psi}_\varepsilon(x) = \frac{1}{|x|^{N-2s}} \psi_\varepsilon\left(\frac{x}{|x|^2}\right), \quad x \in \Omega_\varepsilon \setminus \{0\}.$$

Let Ω_ε^* be the image Ω_ε under the Kelvin transform. Since

$(-\Delta)^s \hat{\psi}_\varepsilon(x) = \frac{1}{|x|^{N+2s}} (-\Delta)^s \psi_\varepsilon\left(\frac{x}{|x|^2}\right)$, doing a straight forward computation we obtain,

$$(4.12) \quad \begin{cases} (-\Delta)^s \hat{\psi}_\varepsilon = \frac{1}{|x|^{4s}} \left(c_\varepsilon^1\left(\frac{x}{|x|^2}\right) - \varepsilon c_\varepsilon^2\left(\frac{x}{|x|^2}\right) \right) \hat{\psi}_\varepsilon & \text{in } \Omega_\varepsilon^*, \\ \hat{\psi}_\varepsilon = 0 & \text{in } \mathbb{R}^N \setminus \Omega_\varepsilon^*. \end{cases}$$

We set,

$$a_\varepsilon(x) := \frac{1}{|x|^{4s}} \left(c_\varepsilon^1\left(\frac{x}{|x|^2}\right) - \varepsilon c_\varepsilon^2\left(\frac{x}{|x|^2}\right) \right).$$

Thus, (4.12) reduces to

$$(-\Delta)^s \hat{\psi}_\varepsilon = a_\varepsilon(x) \hat{\psi}_\varepsilon \quad \text{in } \Omega_\varepsilon^*.$$

Claim: For $N > 4s$, the function $a_\varepsilon \in L^t(\Omega_\varepsilon^*)$, for some $t > \frac{N}{2s}$.

Assuming the claim, let us first complete the proof of step 3. Thanks to the above claim, using Moser iteration technique in the spirit of the proof of [4, Theorem 1.1] (see also [33] and [32, Lemma B.3]), it can be shown that

$$\sup_{\Omega_\varepsilon^* \cap B_1(0)} |\hat{\psi}_\varepsilon| \leq C \left(\int_{\Omega_\varepsilon^* \cap B_2(0)} |\hat{\psi}_\varepsilon|^{2^*} \right)^{\frac{1}{2^*}}.$$

Moreover,

$$\int_{\Omega_\varepsilon^* \cap B_2(0)} |\hat{\psi}_\varepsilon|^{2^*} \leq \int_{\Omega_\varepsilon^*} |\hat{\psi}_\varepsilon|^{2^*} = \int_{\Omega_\varepsilon} |\psi_\varepsilon|^{2^*} \leq C.$$

The last inequality is due to (4.10). Hence $\sup_{\Omega_\varepsilon^* \cap B_1(0)} |\hat{\psi}_\varepsilon| \leq C$. This in turn implies,

$$|\psi_\varepsilon(x)| \leq \frac{C}{|x|^{N-2s}}, \quad x \in \Omega_\varepsilon \setminus B_r(0),$$

for $\varepsilon > 0$ small enough and for some constant $C > 0$ and $r > 0$.

Now, let us prove the claim.

Using (4.4), it is easy to see that $\frac{1}{|x|^{4s}} c_\varepsilon^1\left(\frac{x}{|x|^2}\right) \leq \frac{C}{(\mu_{N,s}^{-1} + |x|^2)^{2s}}$. Hence for $t > \frac{N}{2s}$,

$$(4.13) \quad \int_{\Omega_\varepsilon^*} \frac{1}{|x|^{4st}} c_\varepsilon^1\left(\frac{x}{|x|^2}\right)^t dx \leq C \int_{\mathbb{R}^N} \frac{dx}{(\mu_{N,s}^{-1} + |x|^2)^{2st}} < \infty.$$

On the other hand, $\|\frac{u_\varepsilon(\gamma_\varepsilon x)}{\|u_\varepsilon\|_{L^\infty(\Omega)}}\|_{L^\infty(\Omega)} \leq 1$ implies $|\varepsilon c_\varepsilon^2| \leq q\varepsilon\gamma_\varepsilon^{\frac{(N+2s)-q(N-2s)}{2}}$. Note that, boundedness of Ω implies there exists $R > 0$ such that $\Omega \subseteq B_R(0)$. Hence $\Omega_\varepsilon \subseteq B_{\frac{R}{\gamma_\varepsilon}}(0)$ and $\Omega_\varepsilon^* \subseteq \mathbb{R}^N \setminus B_{\frac{\gamma_\varepsilon}{R}}(0)$. Therefore,

$$\begin{aligned} \int_{\Omega_\varepsilon^*} \frac{1}{|x|^{4st}} c_\varepsilon^2 \left(\frac{x}{|x|^2}\right)^t dx &\leq C \left[\varepsilon \gamma_\varepsilon^{\frac{(N+2s)-q(N-2s)}{2}} \right]^t \int_{\Omega_\varepsilon^*} \frac{dx}{|x|^{4st}} \\ (4.14) \qquad \qquad \qquad &\leq C \left[\varepsilon \gamma_\varepsilon^{\frac{(N+2s)-q(N-2s)}{2}} \right]^t \left(\frac{\gamma_\varepsilon}{R}\right)^{N-4st} \end{aligned}$$

Since $p = 2^* - 1$, from Theorem 1.2, it follows that $\varepsilon \|u_\varepsilon\|^{\frac{q(N-2s)+N-6s}{N-2s}} = C'$, that is, $\varepsilon \gamma_\varepsilon^{-\frac{N-6s+q(N-2s)}{2}} = C'$. As a result,

$$(4.15) \qquad \text{RHS of (4.14)} \leq C \gamma_\varepsilon^{t(N-6s)+N}.$$

Clearly, $N \geq 6s$ implies $\gamma_\varepsilon^{t(N-6s)+N} < C$ for some constant $C > 0$. If $4s < N < 6s$, then choose $t \in (\frac{N}{2s}, \frac{N}{6s-N})$ to get $t(N-6s) + N \geq 0$.

Hence, combining (4.13) and (4.15) the claim follows.

Step 4: Thanks to [16, Theorem 1.1], the linear space of solutions to equation (4.6) can be spanned by the following $(N+1)$ functions:

$$\psi_i(x) = \frac{2x_i}{\left(1 + \frac{|x|^2}{\mu_{N,s}}\right)^{\frac{N-2s+2}{2}}}, \quad i = 1, \dots, N$$

and

$$\psi_{N+1}(x) = \frac{1 - |x|^2}{\left(1 + \frac{|x|^2}{\mu_{N,s}}\right)^{\frac{N-2s+2}{2}}}.$$

That is, general solution of (4.6) can be written as

$$\psi(x) = \alpha \frac{1 - |x|^2}{\left(1 + \frac{|x|^2}{\mu_{N,s}}\right)^{\frac{N-2s+2}{2}}} + \sum_{i=1}^N \beta_i \frac{2x_i}{\left(1 + \frac{|x|^2}{\mu_{N,s}}\right)^{\frac{N-2s+2}{2}}},$$

where $\alpha, \beta_i \in \mathbb{R}$. Since ψ is symmetric function, each $\beta_i = 0$.

Step 5: In this step we will prove that $\alpha = 0$.

Suppose $\alpha \neq 0$. We aim to get a contradiction. For simplicity of the calculation, we can take $\alpha = 1$ and $\mu_{N,s} = 1$, that is,

$$(4.16) \qquad \psi(x) = \frac{1 - |x|^2}{(1 + |x|^2)^{\frac{N-2s+2}{2}}}.$$

Let Ω' be any neighbourhood of $\partial\Omega$, not containing the origin.

$$\textbf{Claim: } \|u_\varepsilon\|_{L^\infty(\Omega)}^2 \frac{(u_\varepsilon(x) - v_\varepsilon(x))}{\|u_\varepsilon - v_\varepsilon\|_{L^\infty(\Omega)} \delta(x)^s} \rightarrow -c_0 \frac{G(x,0)}{\delta(x)^s} \quad \text{uniformly in } \Omega',$$

for some constant $c_0 > 0$.

Indeed,

$$\begin{aligned}
 (-\Delta)^s \left(\|u_\varepsilon\|_{L^\infty(\Omega)}^2 \frac{(u_\varepsilon(x) - v_\varepsilon(x))}{\|u_\varepsilon - v_\varepsilon\|_{L^\infty(\Omega)}} \right) &= \frac{\|u_\varepsilon\|_{L^\infty(\Omega)}^2}{\|u_\varepsilon - v_\varepsilon\|_{L^\infty(\Omega)}} \left[(u_\varepsilon^p - v_\varepsilon^p) - \varepsilon(u_\varepsilon^q - v_\varepsilon^q) \right] \\
 &= \frac{\|u_\varepsilon\|_{L^\infty(\Omega)}^2}{\|u_\varepsilon - v_\varepsilon\|_{L^\infty(\Omega)}} (d_\varepsilon^1(x) - \varepsilon d_\varepsilon^2(x)) (u_\varepsilon - v_\varepsilon) \\
 (4.17) \qquad \qquad \qquad &= : f_\varepsilon,
 \end{aligned}$$

where

$$d_\varepsilon^1(x) = p \int_0^1 (tu_\varepsilon(x) + (1-t)v_\varepsilon(x))^{p-1} dt$$

and

$$d_\varepsilon^2(x) = q \int_0^1 (tu_\varepsilon(x) + (1-t)v_\varepsilon(x))^{q-1} dt.$$

Note that

$$d_\varepsilon^1(\gamma_\varepsilon x) = \gamma_\varepsilon^{-2s} c_\varepsilon^1(x) \quad \text{and} \quad d_\varepsilon^2(\gamma_\varepsilon x) = \gamma_\varepsilon^{-2s} c_\varepsilon^2(x).$$

Therefore, using (4.4), we have

$$(4.18) \qquad d_\varepsilon^1(x) \leq C \gamma_\varepsilon^{-2s} \frac{1}{(\mu_{N,s} + |\frac{x}{\gamma_\varepsilon}|^2)^{2s}} \leq C \frac{\gamma_\varepsilon^{2s}}{|x|^{4s}}.$$

$$(4.19) \qquad d_\varepsilon^2(x) \leq C \frac{\gamma_\varepsilon^{-2s}}{(\mu_{N,s} + |\frac{x}{\gamma_\varepsilon}|^2)^{\frac{(N-2s)(q-1)}{2}}} \leq C \frac{\gamma_\varepsilon^{q(N-2s)-N}}{|x|^{(N-2s)(q-1)}}.$$

Subclaim 1: $\lim_{\varepsilon \rightarrow 0} f_\varepsilon(x) = 0 \quad \forall x \in \Omega'.$

As $\gamma_\varepsilon \rightarrow 0$, using (4.11), (4.18) and (4.19), for $x \in \Omega'$ we obtain

$$\begin{aligned}
 f_\varepsilon(x) &= \|u_\varepsilon\|_{L^\infty(\Omega)}^2 \psi_\varepsilon \left(\frac{x}{\gamma_\varepsilon} \right) (d_\varepsilon^1(x) - \varepsilon d_\varepsilon^2(x)) \\
 &\leq C \|u_\varepsilon\|_{L^\infty(\Omega)}^2 \frac{1}{|\frac{x}{\gamma_\varepsilon}|^{N-2s}} (d_\varepsilon^1(x) + \varepsilon d_\varepsilon^2(x)) \\
 &\leq \frac{C}{|x|^{N-2s}} \left(\frac{\gamma_\varepsilon^{2s}}{|x|^{4s}} + \frac{\gamma_\varepsilon^{q(N-2s)-N}}{|x|^{(N-2s)(q-1)}} \right) \\
 &\rightarrow 0,
 \end{aligned}$$

since $q > \frac{N+2s}{N-2s}$.

Subclaim 2: $\lim_{\varepsilon \rightarrow 0} \int_\Omega f_\varepsilon(x) dx = -c_0$, for some constant $c_0 > 0$.

To see this,

$$\begin{aligned}
 \int_\Omega f_\varepsilon(x) dx &= \frac{\|u_\varepsilon\|_{L^\infty(\Omega)}^2}{\|u_\varepsilon - v_\varepsilon\|_{L^\infty(\Omega)}} \int_\Omega d_\varepsilon^1(x) (u_\varepsilon - v_\varepsilon) dx \\
 &\quad - \frac{\|u_\varepsilon\|_{L^\infty(\Omega)}^2}{\|u_\varepsilon - v_\varepsilon\|_{L^\infty(\Omega)}} \int_\Omega \varepsilon d_\varepsilon^2(x) (u_\varepsilon - v_\varepsilon) dx \\
 &= \int_{\Omega_\varepsilon} c_\varepsilon^1(y) \psi_\varepsilon(y) dy - \varepsilon \int_{\Omega_\varepsilon} c_\varepsilon^2(y) \psi_\varepsilon(y) dy.
 \end{aligned}$$

In the last step, we have used the change of variable $x = \gamma_\varepsilon y$. Using (4.5) and (4.16) via dominated convergence theorem, we obtain

$$(4.20) \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} c_\varepsilon^1(y) \psi_\varepsilon(y) dy = p \int_{\mathbb{R}^N} \frac{1 - |x|^2}{(1 + |x|^2)^{2s + \frac{N-2s+2}{2}}} dx.$$

Using change of variable the RHS of the above equality can be computed as follows:

$$(4.21) \quad \begin{aligned} \int_{\mathbb{R}^N} \frac{1 - |x|^2}{(1 + |x|^2)^{\frac{N+2s+2}{2}}} dx &= \omega_N \int_0^1 \frac{(1 - r^2) r^{N-1}}{(1 + r^2)^{\frac{N+2s+2}{2}}} dr \\ &- \omega_N \int_1^0 \frac{1 - \frac{1}{t^2}}{(1 + \frac{1}{t^2})^{\frac{N+2s+2}{2}}} t^{-2-(N-1)} dt \\ &= -\omega_N \int_0^1 \frac{r^{2s-1} (1 - r^2) (1 - r^{N-2s})}{(1 + r^2)^{\frac{N+2s+2}{2}}} dr \end{aligned}$$

As $s > 0$, $\int_0^1 \frac{r^{2s-1} (1 - r^2) (1 - r^{N-2s})}{(1 + r^2)^{\frac{N+2s+2}{2}}} dr \leq \int_0^1 r^{2s-1} dr < \infty$. Hence from (4.20), we get

$$(4.22) \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} c_\varepsilon^1(y) \psi_\varepsilon(y) dy = -c_0,$$

for some $c_0 > 0$. Similarly it can be shown that

$$|\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} c_\varepsilon^2(y) \psi_\varepsilon(y) dy| < \infty.$$

Therefore,

$$(4.23) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Omega_\varepsilon} c_\varepsilon^2(y) \psi_\varepsilon(y) dy = 0.$$

Combining (4.22) and (4.23), Subclaim 2 follows.

Now we get back to (4.17). Define,

$$\phi_\varepsilon(x) := \|u_\varepsilon\|_{L^\infty(\Omega)}^2 \frac{(u_\varepsilon(x) - v_\varepsilon(x))}{\|u_\varepsilon - v_\varepsilon\|_{L^\infty(\Omega)}}.$$

Then ϕ_ε satisfies

$$\begin{cases} (-\Delta)^s \phi_\varepsilon = f_\varepsilon & \text{in } \Omega, \\ \phi_\varepsilon = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then for any $r > 0$ small and $x \in \Omega'$, we have

$$(4.24) \quad \begin{aligned} \frac{\phi_\varepsilon(x)}{d^s(x)} &= \int_{\Omega} \frac{G(x, y) f_\varepsilon(y)}{d^s(x)} dy \\ &= \int_{B_r(0)} \frac{G(x, y) f_\varepsilon(y)}{d^s(x)} dy + \int_{\Omega \setminus B_r(0)} \frac{G(x, y) f_\varepsilon(y)}{d^s(x)} dy. \end{aligned}$$

Using Lemma 3.5 and Subclaim 1, we estimate the 2nd term on RHS as follows:

$$(4.25) \quad \left| \int_{\Omega \setminus B_r(0)} \frac{G(x, y) f_\varepsilon(y)}{d^s(x)} dy \right| \leq C \int_{\Omega \setminus B_r(0)} \frac{|f_\varepsilon(y)|}{|x - y|^{N-s}} dy = o_{\varepsilon, r}(1),$$

where $o_{r,\varepsilon}(1)$ denote the term going to 0 as $r \rightarrow 0$ and $\varepsilon \rightarrow 0$. Note that we have used the fact that $|x - y|^{s-N}$ is integrable in Ω . Furthermore $\frac{G(x,\cdot)}{d(x)^s}$ is continuous in $\overline{\Omega} \setminus \{x\}$, (see [9, Lemma 6.5]). Therefore from (4.24), we obtain

$$\lim_{\varepsilon \rightarrow 0} \frac{\phi_\varepsilon(x)}{d^s(x)} = \frac{G(x,0)}{d^s(x)} \lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_{B_r(0)} f_\varepsilon(y) dy.$$

Moreover, by Subclaim 2,

$$\lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{G(x,0)}{d^s(x)} \int_{B_r(0)} f_\varepsilon(y) dy = -c_0 \frac{G(x,0)}{d^s(x)}.$$

Thus, it follows

$$(4.26) \quad \lim_{\varepsilon \rightarrow 0} \frac{\phi_\varepsilon(x)}{d^s(x)} = -c_0 \frac{G(x,0)}{d^s(x)}.$$

This proves the claim.

In order to complete the proof, we apply the Pohozaev identity (1.20) to u_ε and v_ε .

$$\begin{aligned} \Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{u_\varepsilon(x)}{d^s(x)} \right)^2 (x \cdot \nu) dS &= \varepsilon \left[(N-2s) - \frac{2N}{q+1} \right] \int_{\Omega} u_\varepsilon^{q+1} dx, \\ \Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{v_\varepsilon(x)}{d^s(x)} \right)^2 (x \cdot \nu) dS &= \varepsilon \left[(N-2s) - \frac{2N}{q+1} \right] \int_{\Omega} v_\varepsilon^{q+1} dx. \end{aligned}$$

Subtracting one from the other and multiplying by $\frac{\|u_\varepsilon\|_{L^\infty(\Omega)}^3}{\|u_\varepsilon - v_\varepsilon\|_{L^\infty(\Omega)}}$ in both sides yields,

$$\begin{aligned} (4.27) \quad & \Gamma(1+s)^2 \int_{\partial\Omega} \frac{\|u_\varepsilon\|_{L^\infty(\Omega)}^2 (u_\varepsilon - v_\varepsilon)}{\|u_\varepsilon - v_\varepsilon\|_{L^\infty(\Omega)} d^s(x)} \frac{(u_\varepsilon + v_\varepsilon) \|u_\varepsilon\|_{L^\infty(\Omega)}}{d^s(x)} (x \cdot \nu) dS \\ &= \varepsilon \left[(N-2s) - \frac{2N}{q+1} \right] (q+1) \int_{\Omega} \frac{\|u_\varepsilon\|_{L^\infty(\Omega)}^3}{\|u_\varepsilon - v_\varepsilon\|_{L^\infty(\Omega)}} (u_\varepsilon - v_\varepsilon) \int_0^1 (tu_\varepsilon + (1-t)v_\varepsilon)^q dt dx. \end{aligned}$$

By doing the change of variable $x = \gamma_\varepsilon y$, RHS of (4.27) reduces as

$$\begin{aligned} \text{RHS of (4.27)} &= \varepsilon \|u_\varepsilon\|^{q-p+2} [q(N-2s) - (N+2s)] \\ &\quad \times \int_{\Omega_\varepsilon} \psi_\varepsilon(y) \left[\int_0^1 \left(t \frac{u_\varepsilon(\gamma_\varepsilon y)}{\|u_\varepsilon\|_{L^\infty(\Omega)}} + (1-t) \frac{v_\varepsilon(\gamma_\varepsilon y)}{\|v_\varepsilon\|_{L^\infty(\Omega)}} \right)^q dt \right] dy. \end{aligned}$$

Note that By Theorem 1.2, $\lim_{\varepsilon \rightarrow 0} \varepsilon \|u_\varepsilon\|^{q-p+2} [q(N-2s) - (N+2s)] = C_1$, for some constant $C_1 > 0$. Therefore, using dominated convergence theorem via (4.4) and (4.16), we obtain

$$(4.28) \quad \lim_{\varepsilon \rightarrow 0} \text{RHS of (4.27)} = C_1 \int_{\mathbb{R}^N} \frac{1 - |x|^2}{(1 + |x|^2)^{\frac{N-2s+2+q(N-2s)}{2}}} dx.$$

Applying the change of variable as in (4.21), it can be proved that

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{1 - |x|^2}{(1 + |x|^2)^{\frac{N-2s+2+q(N-2s)}{2}}} dx &= \omega_N \int_0^1 \frac{r^{N-1} (1-r^2) (1-r^{q(N-2s)-(N+2s)})}{(1+r^2)^{\frac{(N-2s)(q+1)}{2}+1}} dr \\ &= C_2, \end{aligned}$$

where $C_2 > 0$ is a constant. Hence,

$$(4.29) \quad \lim_{\varepsilon \rightarrow 0} [\text{RHS of (4.27)}] > 0.$$

On the other hand, applying (1.24) and (4.26) to LHS via dominated convergence theorem, we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} [\text{LHS of (4.27)}] &= 2\Gamma(1+s)^2 \int_{\partial\Omega} -c_0 \frac{G(x,0)}{d^s(x)} \frac{\omega_N c_{N,s}^{2^*-1}}{2} \frac{\Gamma(\frac{N}{2})\Gamma(s)}{\Gamma(\frac{N+2s}{2})} \frac{G(x,0)}{d^s(x)} (x \cdot \nu) dS \\ &= -\frac{c_0 \omega_N c_{N,s}^{2^*-1}}{2} \frac{\Gamma(\frac{N}{2})\Gamma(s)\Gamma(1+s)^2}{\Gamma(\frac{N+2s}{2})} \int_{\partial\Omega} \left(\frac{G(x,0)}{d^s(x)} \right)^2 (x \cdot \nu) dS \\ (4.30) \quad &< 0. \end{aligned}$$

Combining (4.29) along with (4.30) gives the contradiction. Hence the claim follows.

Step 6: Step 5 implies that $\psi \equiv 0$. Therefore, by Step 1, $\psi_\varepsilon \rightarrow 0$ in K for every compact set K in \mathbb{R}^N . Let $y_\varepsilon \in \mathbb{R}^N$ such that

$$\psi_\varepsilon(y_\varepsilon) = \|\psi_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1.$$

This in turn implies $y_\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$. On the other hand, by (4.11), we get $\psi_\varepsilon(y_\varepsilon) \rightarrow 0$. This contradicts the fact that $\psi_\varepsilon(y_\varepsilon) = 1$. Hence the uniqueness result follows. \square

APPENDIX A.

Define

$$(1.1) \quad \hat{F}(w) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} dx dy + \frac{1}{q+1} \int w^{q+1} dx,$$

where $q > p \geq 2^* - 1$. For $\rho > 0$, set

$$\begin{aligned} X_0(\rho\Omega) &:= \{w \in H^s(\mathbb{R}^N) : w = 0 \text{ in } \mathbb{R}^N \setminus \rho\Omega\}, \\ N_\rho &= \left\{ w \in X_0(\rho\Omega) \cap L^{q+1}(\rho\Omega) : \int_{\rho\Omega} w^{p+1} dx = 1 \right\}. \end{aligned}$$

Define

$$S_\rho := \inf_{w \in N_\rho} \hat{F}(w).$$

Theorem A.1. (i) If $p = 2^* - 1$, then $S_\rho \rightarrow \frac{S}{2}$ as $\rho \rightarrow \infty$, where S is as defined in (1.10).

(ii) If $p > 2^* - 1$, then $S_\rho \rightarrow \mathcal{K}$ as $\rho \rightarrow \infty$, where \mathcal{K} is as defined in (1.8).

Proof. **Step 1:** First we prove that $\lim_{\rho \rightarrow \infty} S_\rho \leq \frac{S}{2}$. Let us consider the function $U(x)$ defined as in (1.11). We know that S is achieved by U and U is the unique ground state solution of (1.13) with $\int_{\mathbb{R}^N} U^{2^*}(x) dx = 1$.

Define

$$U_\rho(x) := \rho^{-\frac{(N-2s)}{4}} U\left(\frac{x}{\sqrt{\rho}}\right) \quad \text{and} \quad \phi_\rho(x) = \phi\left(\frac{x}{\rho}\right)$$

where $\phi \in C_0^\infty(\mathbb{R}^N)$, $\text{supp}(\phi) \subset \Omega$, $\phi \equiv 1$ in $\frac{\Omega}{2}$, $0 \leq \phi \leq 1$, $|\nabla \phi| \leq \frac{2}{d}$, where $d = \text{diam}(\Omega)$. It is easy to see that U_ρ is also a solution of (1.13).

Set $v_\rho(x) := U_\rho(x)\phi_\rho(x)$ and $\hat{v}_\rho(x) = \frac{v_\rho}{|v_\rho|_{L^{2^*}(\rho\Omega)}}$. Then $\hat{v}_\rho \in N_\rho$ and thus,

$$(1.2) \quad S_\rho \leq \hat{F}(\hat{v}_\rho)$$

Note that,

$$\int_{\rho\Omega} v_\rho^{2^*} dx = \rho^{-\frac{N}{2}} \int_{\rho\Omega} U^{2^*}(\frac{x}{\sqrt{\rho}}) \phi^{2^*}(\frac{x}{\rho}) dx = \int_{\rho\Omega} U^{2^*}(x) \phi^{2^*}(\frac{x}{\sqrt{\rho}}) dx.$$

Therefore,

$$(1.3) \quad \lim_{\rho \rightarrow \infty} \int_{\rho\Omega} v_\rho^{2^*} dx = \int_{\mathbb{R}^N} U^{2^*}(x) dx = 1.$$

Similarly,

$$(1.4) \quad \lim_{\rho \rightarrow \infty} \int_{\rho\Omega} \hat{v}_\rho^{q+1} dx = \lim_{\rho \rightarrow \infty} \frac{\rho^{\frac{(N+2s)-q(N-2s)}{4}}}{|v_\rho|_{L^{2^*}(\rho\Omega)}^{q+1}} \int_{\mathbb{R}^N} U^{q+1}(x) \phi^{q+1}(\frac{x}{\sqrt{\rho}}) dx = 0,$$

as $q > \frac{N+2s}{N-2s}$. Hence, from (1.2),

$$(1.5) \quad \lim_{\rho \rightarrow \infty} S_\rho \leq \lim_{\rho \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\hat{v}_\rho(x) - \hat{v}_\rho(y)|^2}{|x - y|^{N+2s}} dx dy = \lim_{\rho \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_\rho(x) - v_\rho(y)|^2}{|x - y|^{N+2s}} dx dy.$$

Now,

$$(1.6) \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_\rho(x) - v_\rho(y)|^2}{|x - y|^{N+2s}} dx dy = I_\rho^1 + I_\rho^2 + I_\rho^3,$$

where

$$\begin{aligned} I_\rho^1 &:= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|U_\rho(x) - U_\rho(y)|^2}{|x - y|^{N+2s}} \phi_\rho^2(x) dy dx, \\ I_\rho^2 &:= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi_\rho(x) - \phi_\rho(y)|^2}{|x - y|^{N+2s}} U_\rho^2(y) dx dy, \\ I_\rho^3 &:= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(U_\rho(x) - U_\rho(y))(\phi_\rho(x) - \phi_\rho(y))U_\rho(y)\phi_\rho(x)}{|x - y|^{N+2s}} dx dy. \end{aligned}$$

A simple calculation yields

$$\begin{aligned} \lim_{\rho \rightarrow \infty} I_\rho^1 &= \lim_{\rho \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|U(x) - U(y)|^2}{|x - y|^{N+2s}} \phi^2(\frac{x}{\sqrt{\rho}}) dy dx \\ (1.7) \quad &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|U(x) - U(y)|^2}{|x - y|^{N+2s}} dy dx = S. \end{aligned}$$

Using change of variable, it is not difficult to see that

$$(1.8) \quad I_\rho^2 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} F_\rho(x, y) dx dy, \quad \text{where} \quad F_\rho(x, y) = \frac{|\phi(\frac{x}{\sqrt{\rho}}) - \phi(\frac{y}{\sqrt{\rho}})|^2 U^2(x)}{|x - y|^{N+2s}}.$$

Clearly, $F_\rho(x, y) \rightarrow 0$ pointwise as $\rho \rightarrow \infty$. Using dominated convergence theorem, we aim to show that $\lim_{\rho \rightarrow \infty} I_\rho^2 = 0$. Let

$$D_1 := \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |x - y| \leq 1\},$$

$$D_2 := \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |x - y| > 1\}.$$

Thus,

$$I_\rho^2 = \int_{D_1} F_\rho(x, y) dx dy + \int_{D_2} F_\rho(x, y) dx dy =: I_\rho^{2,1} + I_\rho^{2,2}$$

In D_1 , we estimate $F_\rho(x, y)$ as follows:

$$\begin{aligned} F_\rho(x, y) &= \frac{|\phi(\frac{x}{\sqrt{\rho}}) - \phi(\frac{y}{\sqrt{\rho}})|^2 U^2(x)}{|x - y|^{N+2s}} \leq \frac{|\frac{x}{\sqrt{\rho}} - \frac{y}{\sqrt{\rho}}|^2 \|\nabla \phi\|_{L^\infty(\mathbb{R}^N)} U^2(x)}{|x - y|^{N+2s}} \\ &\leq \frac{1}{\rho} |x - y|^{2-(N+2s)} \|\nabla \phi\|_{L^\infty(\mathbb{R}^N)} U^2(x) \\ &\leq |x - y|^{2-(N+2s)} \|\nabla \phi\|_{L^\infty(\mathbb{R}^N)} U^2(x), \end{aligned}$$

for $\rho > 1$. Moreover,

$$\begin{aligned} &\int_{D_1} |x - y|^{2-(N+2s)} \|\nabla \phi\|_{L^\infty(\mathbb{R}^N)} U^2(x) dy dx \\ &\leq \|\nabla \phi\|_{L^\infty(\mathbb{R}^N)} \int_{x \in \mathbb{R}^N} U^2(x) \int_{y \in \mathbb{R}^N, |x-y| \leq 1} |x - y|^{2-(N+2s)} dy dx \\ &= \|\nabla \phi\|_{L^\infty(\mathbb{R}^N)} \|U\|_{L^2(\mathbb{R}^N)}^2 N w_N \int_0^1 r^{1-2s} dr < \infty. \end{aligned}$$

Hence, by the dominated convergence theorem we see that $\lim_{\rho \rightarrow \infty} I_\rho^{2,1} = 0$. On the other hand, in D_2 we estimate $F_\rho(x, y)$ as follows:

$$(1.9) \quad F_\rho(x, y) \leq \frac{4 \|\phi\|_{L^\infty(\mathbb{R}^N)} U^2(x)}{|x - y|^{N+2s}}.$$

Proceeding same way as above, we can show that RHS of (1.9) is in $L^\infty(D_2)$. Hence, by the dominated convergence theorem we see that $\lim_{\rho \rightarrow \infty} I_\rho^{2,2} = 0$. Consequently,

$$(1.10) \quad \lim_{\rho \rightarrow \infty} I_\rho^2 = 0.$$

Using change of variable, we see that

$$(1.11) \quad I_\rho^3 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} H_\rho(x, y) dx dy,$$

where,

$$H_\rho(x, y) = \frac{|U(x) - U(y)| |\phi(\frac{x}{\sqrt{\rho}}) - \phi(\frac{y}{\sqrt{\rho}})| |U(x)| |\phi(\frac{y}{\sqrt{\rho}})|}{|x - y|^{N+2s}}.$$

Clearly $H_\rho(x, y) \rightarrow 0$ pointwise as $\rho \rightarrow \infty$. Moreover,

$$\begin{aligned} |H_\rho(x, y)| &\leq \frac{|U(x) - U(y)| |\phi(\frac{x}{\sqrt{\rho}}) - \phi(\frac{y}{\sqrt{\rho}})| |U(x)| |\phi(\frac{y}{\sqrt{\rho}})|}{|x - y|^{N+2s}} \\ (1.12) \quad &\leq \frac{1}{2} \frac{|U(x) - U(y)|^2}{|x - y|^{N+2s}} + \frac{1}{2} \frac{|\phi(\frac{x}{\sqrt{\rho}}) - \phi(\frac{y}{\sqrt{\rho}})|^2 U^2(x)}{|x - y|^{N+2s}} \end{aligned}$$

The 1st term on RHS is in $L^1(\mathbb{R}^N \times \mathbb{R}^N)$ and 2nd term can be dominated by L^1 function as before. Hence by dominated convergence theorem, we have

$$(1.13) \quad \lim_{\rho \rightarrow \infty} I_\rho^3(x, y) = \lim_{\rho \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} H_\rho(x, y) dx dy = 0.$$

As a result, combining (1.7), (1.10), (1.13), along with (1.6) and (1.5) we obtain Hence we obtain that $\lim_{\rho \rightarrow \infty} S_\rho \leq \frac{\mathcal{S}}{2}$.

Step 2: In this step we aim to show $S_\rho \geq \frac{S}{2}$. let $\delta > 0$ be arbitrary. As $S_\rho = \inf_{w \in N_\rho} \hat{F}(w)$, there exists $w_{\rho,\delta} \in N_\rho$ such that

$$(1.14) \quad \hat{F}(w_{\rho,\delta}) < S_\rho + \delta.$$

Let $\eta(\cdot)$ be the standard mollifier function, i.e, $\eta(x) = C \exp(\frac{1}{|x|^2-1})$ if $|x| < 1$ and 0 otherwise. Set $\eta_\sigma(x) = \sigma^{-N} \eta(\frac{x}{\sigma})$.

Define $w_{\rho,\delta}^\sigma := w_{\rho,\delta} * \eta_\sigma$ and $v_{\rho,\delta}^\sigma = \frac{w_{\rho,\delta}^\sigma}{|w_{\rho,\delta}^\sigma|_{L^{2^*}(\mathbb{R}^N)}}$.

We note that $v_{\rho,\delta}^\sigma \in C_0^\infty(\mathbb{R}^N) \cap N$ where

$$N := \{w \in D^{s,2}(\mathbb{R}^N) : w \in L^{q+1}(\mathbb{R}^N), \int_{\mathbb{R}^N} w^{2^*} dx = 1\}$$

and $D^{s,2}(\mathbb{R}^N)$ is completion of $C_0^\infty(\mathbb{R}^N)$ with the norm $\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}$.

Note that $v_{\rho,\delta}^\sigma \rightarrow w_{\rho,\delta}$ in $D^{s,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$ as $\sigma \rightarrow 0$.

Hence, we have,

$$\frac{S}{2} \leq \hat{F}(v_{\rho,\delta}^\sigma) \rightarrow \hat{F}(w_{\rho,\delta}).$$

Combining this with (1.14) we have, $\frac{S}{2} < S_\rho + \delta$. As $\delta > 0$ is arbitrary we have, $\frac{S}{2} \leq S_\rho$. This implies, $\frac{S}{2} \leq \lim_{\rho \rightarrow \infty} S_\rho$. This completes the proof.

Second part:

Let $w \in D^{s,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$ be a minimizer for \mathcal{K} (existence is guaranted by [4, Theorem 1.4]) with $\int_{\mathbb{R}^N} w^{p+1} dx = 1$.

Define ϕ_ρ as in step 1. Set $w_\rho := w\phi_\rho$ and $\hat{w}_\rho = \frac{w_\rho}{|w_\rho|_{L^{p+1}(\mathbb{R}^N)}}$. Then $\hat{w}_\rho \in N_\rho$. Consequently, $S_\rho \leq \hat{F}(\hat{w}_\rho)$. Proceeding before as in step 1, we can show that $\hat{F}(\hat{w}_\rho) \rightarrow \mathcal{K}$ as $\rho \rightarrow \infty$. Hence, $\lim_{\rho \rightarrow \infty} S_\rho \leq \mathcal{K}$. To get the other sided inequality, we use the same idea as first part. Hence, the result follows. \square

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